MAY 8 1967

Recd SS 5/8/67

INSTITUTE FOR SPACE STUDIES

SG

RELATIVITY THEORY AND ASTROPHYSICS

Hong-Yee Chiu

N71-72185
(ACCESSION NUMBER) (THRU)
276 None
(PAGES) (CODE)
TMX-66978
(NASA CR OR TMX OR AD NUMBER) (CATEGORY)
FACILITY FORM 602

RECEIVED
NASA STI FACILITY
INPUT BRANCH
NOV 1970

GODDARD SPACE FLIGHT CENTER
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

RELATIVITY THEORY AND ASTROPHYSICS

Hong-Yee Chiu

Institute for Space Studies,
Goddard Space Flight Center,
National Areonautics and Space Administration
New York, N.Y.

and

Physics Department and
Earth and Space Sciences Department
State University of New York
Stony Brook, New York

February 1967

GENERAL RELATIVITY

Lecture Notes

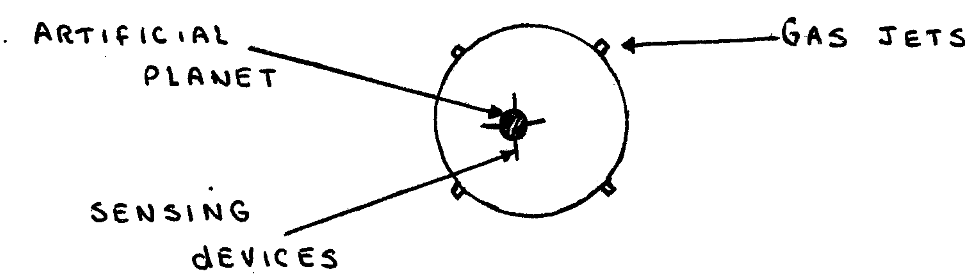
H.Y. CHIU (Notes by the Class)

The Differences between General Relativity and Electrodynamics

Static Classical	Cavendish force $= \frac{mM}{R^2}$	Coulomb force $= e^2/R^2$
	$f \sim 1/R^m$	$f \sim 1/R^m$
	$m = 2 \pm 10^{-6}$	$m = 2 \pm 10^{-6}$
Dynamic Classical	It is not known if gravitational waves exist. DATA - The theory predicts the bending of light rays around the sun and the motion of the perihelion of a planet - both have been observed.	Maxwell's equations predict the existence of electromagnetic waves, and their existence is verified by many experiments.
Quantization	needed ? existence?	Hydrogen spectra - well developed and verified.
Second Quantization	?	Lamb shift - well developed and verified.

The coupling constant for electromagnetic interactions is $\alpha = e^2/\hbar c = 1/137$. We can obtain the gravitational coupling constant by replacing e^2 by the gravitational charge $Gm_p^{\,2}$ and $\alpha_G = Gm_p^{\,2}/\hbar c = 10^{-39}$. For two gravitationally bound neutrons the first Bohr orbit will be 2×10^{18} cm or two light years. Thus, in order to perform gravitational experiments, one must use objects of astronomical size.

An example of a space experiment that might show some relativistic effects is an artificial planet surrounded by some gas shield to eliminate gas drag and light pressure difficulties that would interfere with the gravitational orbit. We surround a small artificial planet with a hollow spherical shell equipped with gas jets as shown.



When the artificial planet gets near the wall, the gas jets are turned on by the sensing devices to compensate the drag to keep the shell from touching the inner sphere. In this way the artificial planet moves on a proper planetary orbit. A radar technique could be used to determine the distance to this device with great precision making possible a test of the form of the force law to the accuracy required for relativistic effects to appear. An alternative and simpler technique would be to set two or three vehicles, externally identical but of different density, on substantially the same initial orbit. Then the effect of gas drag and light pressure on their orbits could be inferred from their different motions.

To verify general relativity we need non-local experiments because the space is locally Euclidean.

Structure of General Relativity and Electrodynamics

	General Relativity	Electrodynamics
Space	Determined by the theory and the mass-energy distribution.	Given element $ds^2 = c^2dt^2 - (dx^2 + dy^2 + dz^2)$
Equations of Motion	Determined by the theory - (non-linear theory)	Given element (not from Maxwell's equations) $m\frac{d^2\vec{r}}{dt^2} = e\vec{E}$ (linear)

When are the relativistic effects important?

Quantum effects become important when

$$p\,r \sim \hbar \; .$$

Relativistic effects become important when the kinetic energy is about equal to $m\,c^2$

$$\frac{m\ G\ M}{R} = \text{kinetic energy} \sim m\ c^2$$

or

$$\frac{G\ M}{R\ c^2} \sim 1$$

Principles of Eqiivalence -

(1) The weak principle of equivalence states that the trajectory of a particle, under the influence of gravitational fields only, depends only in its initial position and velocity, but not on its mass or its nature. If, for example, we consider a simple pendulum with the equation of motion

$$m_i\ \ddot{\theta} = m_g\ \ell\ g\ \theta \qquad ,$$

where m_i is the property of mass in response to acceleration and m_g, is the property of mass in response to gravitation; the weak principle of equivalence states that

$$m_i\ /\ m_g \equiv 1\ .$$

Experiments which have measured this ratio give the following results

	$m_i\ /\ m_g$
Newton	1 ± 10^{-3}
Bessel	1 ± 10^{-4}
Eötvös	1 ± 10^{-8}
Dicke	$1 \pm 5\text{x}10^{-12}$

2) The strong principle of equivalence states that "the laws of physics must be of such a nature that they apply to systems of reference in any kind of motion." Thus at every point in space-time there exists an inertial frame of reference in which all the physical laws are the same as they would be in the absence of gravitation (one can transform away the effects of gravity by allowing the laboratory to fall freely). We also have that the dimensionless constants appearing in the physical laws must be the same throughout the universe.

If the coupling constants were position dependent the binding energy of a body would be position dependent. In this case, a body in a gravitational field would also experience a force depending on position and structure. The results of the Eötvös experiment are accurate enough to make it unlikely that the strong and electromagnetic coupling constants are position dependent, but nothing can be said about the gravitational and weak interaction coupling constants.

Principle of Covariance

The general laws of nature are to be expressed by equations which hold good for all systems of coordinates, that is, are co-variant with respect to any substitution whatever (generally covariant).

This requirement of general covariance exceeds the principle of general relativity which makes reference only to inertial systems. The principle of covariance suggests that the laws of physics have the same form relative to any (not necessarily freely falling) observer.

The principle of covariance thus implies a relativity of gravitational forces.

However, let us consider some distance ds measured in a two dimensional space

$$ds^2 = dx^2 + dy^2 \qquad \text{or} \qquad ds^2 = r^2 \,(d\theta^2 + \sin^2\theta \; d\phi^2)$$

There exists no transformation from cartesian coordinates to polar coordinates which also preserves the distance ds. An example of this is the difficulty of representing the earth's surface on a flat map.

Without a principle of covariance it would be impossible to interpret the results of observations in these two spaces.

Curvature of Space-Time

What do we mean by curvature of "empty" space? The very concept of curved, empty space seems impossible to visualize.

Let us look at a somewhat simpler, although analogous, situation. Suppose we are given a table of distances between cities. Can we determine if these cities lie on a curved surface or a flat plane?

As an example let us consider some airports and the flight distances between these various airports.

	AZORES	BERLIN	BOMBAY	BUENOS AIRES
Azores	—	2148	5930	5385
Berlin	2148	—	3947	7411
Bombay	5930	3947	—	9380
Buenos Aires	5385	7411	9380	—

Using a ruler and compass these points can easily be plotted. Choose a point as representing the Azores. Then scribe a radius scaled to the distance from the Azores to Berlin. Choose any point on this arc as Berlin. The choice of these two airports now completely determines all others in the table since the only variables to be chosen are absolute location and orientation.

Scribe arcs from the Azores and Berlin representing the distance to Bombay. The intersection of these arcs determines Bombay. Now from these three airports scribe distances to Buenos Aires. If the surface of the Earth were flat, all these arcs would meet at a point - Buenos Aires. However, we find that they do not meet.

The first conclusion is that the surface of the Earth (the space in question) is not flat. It seems quite reasonable that a sphere of <u>some</u> radius will be able to fit the distances. For only four points one can always find an appropriate sphere. However, for five or more points a sphere may not be general enough.

The point of this exercise is that the distances between points characterize the space.

1) Geometrical properties of a "space" are characterized by the distances between all points in space.

This is a rather exhaustive specification. Is there not some more tractable method of defining a space?

In the previous example it is obviously not necessary to know the distance from New York to San Francisco if one knows the distances from New York to Chicago, and from Chicago to San Francisco and if one knows the airline route from New York to San Francisco passes through Chicago. Thus it isn't necessary to have so extensive a Table. It is enough to know the distance between every point and <u>all</u> the neighboring points.

The distance from any point to a nearby point depends bi-linearly upon the coordinate differences between the two points -

$$(ds)^2 = \underset{\substack{\uparrow \\ \text{metric coefficients - contains all} \\ \text{information about the space.}}}{g_{\alpha\beta}} \, dx^{\alpha} \, dx^{\beta}$$

What criteria must we satisfy in order to be sure our space is flat? Take a Taylor expansion of $g_{\alpha\beta}$ -

$$g_{\alpha\beta} = g_{\alpha\beta}\left(\{x_o\}\right) + \frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} \Delta x^{\gamma}$$

The second term can be neglected if -

$$\frac{1}{g_{\alpha\beta}} \frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} \Delta x^{\gamma} \ll 1$$

that is, it is always possible to choose a Δx such that we have a Euclidean Space. Although Special Relativity is not a Euclidean system one can always find a local neighborhood of points within which Special Relativity is valid.

Equations of Motion

Suppose we are given the metric tensor (metric coefficients). This should completely determine the equations of motion. We shall determine the equations of a path between two points such that

$$\int ds \quad \text{is stationary} \quad .$$

Keeping the beginning and end of the path fixed, we give every intermediate point some arbitrary displacement δx_σ.

$$ds^2 = g_{\mu\nu}\, dx_\mu\, dx_\nu$$

$$(1) \quad 2ds\,\delta(ds) = dx_\mu\, dx_\nu\, \delta g_{\mu\nu} + g_{\mu\nu}\, dx_\mu\, \delta(dx_\nu) + g_{\mu\nu}\, dx_\nu\, \delta(dx_\mu)$$

$$= dx_\mu\, dx_\nu \frac{\partial g_{\mu\nu}}{\partial x_\sigma}\delta x_\sigma + g_{\mu\nu}\, dx_\mu\, d(\delta x_\nu) + g_{\mu\nu}\, dx_\nu\, d(\delta x_\mu)$$

The stationary condition is

$$\int \delta(ds) = 0$$

Substituting (1)

$$\frac{1}{2}\int\left\{\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\delta x_\sigma + g_{\mu\nu}\frac{dx_\mu}{ds}\frac{d}{ds}(\delta x_\nu) + g_{\mu\nu}\frac{dx_\nu}{ds}\frac{d}{ds}(\delta x_\mu)\right\} ds = 0$$

Changing dummy indices in the last two terms -

$$\frac{1}{2}\int\left\{\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\delta x_\sigma + \left(g_{\mu\sigma}\frac{dx_\mu}{ds} + g_{\sigma\nu}\frac{dx_\nu}{ds}\right)\frac{d}{ds}(\delta x_\sigma)\right\} ds = 0$$

Integrate by parts (note $\delta x_\sigma = 0$ on boundaries)

$$\frac{1}{2}\int\left\{\frac{dx_\mu}{ds}\,\frac{dx_\nu}{ds}\,\frac{\partial g_{\mu\nu}}{\partial x_\sigma} - \frac{d}{ds}\left(g_{\mu\sigma}\frac{dx_\mu}{ds} + g_{\sigma\nu}\frac{dx_\nu}{ds}\right)\right\} ds = 0$$

For arbitrary δx_σ the coefficient in the integral must vanish at all points along the path.

Therefore

$$\frac{1}{2}\,\frac{dx_\mu}{ds}\,\frac{dx_\nu}{ds}\,\frac{\partial g_{\mu\nu}}{\partial x_\sigma} - \frac{1}{2}\,\frac{dg_{\mu\sigma}}{ds}\,\frac{dx_\mu}{ds} - \frac{1}{2}\,\frac{dg_{\sigma\nu}}{ds}\,\frac{dx_\nu}{ds} - \frac{1}{2}\,g_{\mu\sigma}\,\frac{d^2x_\mu}{ds^2} - \frac{1}{2}g_{\sigma\nu}\frac{d^2x_\nu}{ds^2} =$$

But

$$\frac{dg_{\mu\sigma}}{ds} = \frac{\partial g_{\mu\sigma}}{\partial x_\nu}\,\frac{dx_\nu}{ds} \qquad \frac{dg_{\sigma\nu}}{ds} = \frac{\partial g_{\sigma\nu}}{\partial x_\mu}\,\frac{dx_\mu}{ds}$$

Now in the last two terms replace the dummy suffixes μ and ν by ε. Then

$$\frac{1}{2}\,\frac{dx_\mu}{ds}\,\frac{dx_\nu}{ds}\left(\frac{\partial g_{\mu\nu}}{\partial x_\sigma} - \frac{\partial g_{\mu\sigma}}{\partial x_\nu} - \frac{\partial g_{\nu\sigma}}{\partial x_\mu}\right) - g_{\varepsilon\sigma}\,\frac{d^2x_\varepsilon}{ds^2} = 0$$

Multiply through by $g^{\sigma\alpha}$

$$\frac{1}{2}\,\frac{dx_\mu}{ds}\,\frac{dx_\nu}{ds}\,g^{\sigma\alpha}\left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\nu\sigma}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\sigma}\right) + \frac{d^2x_\alpha}{ds^2} = 0$$

$g^{\sigma\alpha}$ is the inverse to $g_{\alpha\beta}$, such that $g^{\sigma\alpha}g_{\alpha\beta} = \delta^\sigma_\beta$.

From the definition of the Christoffel symbol -

$$\begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} = \frac{1}{2} g^{\alpha\beta} \left(\frac{\partial g_{\beta\mu}}{\partial x_\nu} + \frac{\partial g_{\nu\beta}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\beta} \right)$$

$$\frac{d^2x}{ds^2} + \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} \frac{dx}{ds} \frac{dx}{ds} = 0$$

For $\alpha = 1,2,3,4$ this gives the equations of motion.

Notation:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

(Greek sub or superscripts) 0,1,2,3

(Roman sub or superscripts) 1,2,3

x^o - time coordinate x^1, x^2, x^3 - spatial coordinates

signature - refers to locally flat coordinates $ds^2 = (+\ ---)$

natural coordinates - refers to the vicinity of a point; coordinates which move with constant velocity with respect to the point are said to be natural coordinates.

proper coordinates - Near a point the gravitational field can be transformed away by introducing a freely falling coordinate system with respect to the gravitational field. Out of the family of coordinate systems which are freely falling, the one which is temporarily at rest with the point under consideration is called the proper coordinate system. All the other coordinate systems are called natural coordinate systems.

A freely falling, non rotating observer will not see a gravitational field in his immediate vicinity. That is, in a finite four dimensional space time region V (including the observer and his laboratory) there is a limit of experimental accuracy such that no true gravitational field can be detected within V as long as this accuracy is not exceeded.

Let grad Φ_G be a measure of the Riemann curvature within V and let ℓ be a typical dimension of V. Then if grad $\Phi_G \cdot \frac{\ell^2}{c^2} \ll 1$ is too small to be measured, the volume enclosed within V is flat within experimental accuracy.

There is another method of obtaining this result. The curvature (grad Φ_G) gives the acceleration at any point. If we assume a uniform curvature (uniform acceleration) we can determine the total acceleration a "particle" receives as it traverses the laboratory

$$\text{residual acceleration} = \text{grad } \Phi_G \cdot \ell$$

The time of transit of light across the lab is ℓ / c. From the familiar formula for final velocity under uniform acceleration -

$$\text{final velocity} = \text{grad } \phi_G \cdot \ell \frac{\ell}{c} .$$

This final velocity must be much less than the velocity of light.

$$\frac{\ell}{c} \text{ grad } \phi_G \cdot \ell \ll c \Rightarrow \text{grad } \phi_G \frac{\ell^2}{c^2} \ll 1 .$$

As an example consider a "laboratory" whose dimensions are those of an elementary particle.

$$\ell \backsim 10^{-13} \text{ cm} \Rightarrow \ell / c \backsim 10^{23} \text{ seconds}$$

If

$$\text{grad } \phi_G \ll 10^{46} \text{ cm/sec}^2\text{/cm}$$

Then

$$\text{grad } \phi_G \frac{\ell^2}{c^2} \ll 10^{46}\ 10^{-46} \backsim 1$$

and we conclude that the restriction is good for elementary particles. Let us determine how much mass of an elementary particle must have to violate the restriction.

$$\phi_G \backsim \left| \frac{1}{r^2} \right|$$

Consider two elementary particles that are touching. This will present the strongest possible field.

$$\phi_G = \frac{1}{(r_1 + \rho)^2} \qquad r_1 = R \quad \rho = R$$

$$R \sim 10^{-13} \text{ cm.}$$

if $\phi_G \succeq 10^{46}$ the restriction is not valid

$$\phi_G \approx 10^{46} = \frac{GM}{R^2} = \frac{10^{-7} \times 7}{10^{-26}} M$$

$$M \sim 10^{27} \text{ g / particle}$$

If the mass is this large, the restriction doesn't hold. Therefore gravitational effects can be entirely neglected from particle theory.

14.3 Riemannian Geometry.

It is impossible in this section to exhaust all features of Riemannian Geometry which might be relevant to general relativity problems, hence we shall concentrate on certain basic features of Riemannian Geometry or those of wide applications to relativity problems of interest. The problem of relating properties of Riemannian Geometry to Physical problems will be discussed in Sect. (14.4) and (14.9).

(i) Generalized Coordinate Systems. In a space of n dimensions the position of a point P is characterized by a set of n numbers $\left\{ x^i \right\}$. In a rectangular orthodognal coordinate system for an Euclidean space,* it is always possible to find a set of constant vectors $\vec{a}_i$ such that

---

*In what follows we shall use Euclidean and Minkowskian. In the ordinary usage a space is said to Euclidean if the line element ds has the following form:

$$ds^2 = dx_0^{\nu} dx_0^{\nu} = 0$$

and all coordinates are real. In the Minkowskian space we have

$$ds^2 = -(dx^2 + dy^2 + dz^2) + c^2 dt^2 \geq 0$$

We shall use, for our discussion, the following definition for an Eucleidean space: A space is Eucleidean if a coordinate system exists such that

(continued on p. 17)

the position of the point P is given by $x^i \vec{a}_i$. This is not true in general, even in Euclidean space. For example, in a spherical polar coordinate system, no such a set of constant vector $\vec{a}_i$ exists. In general we must regard a point in space of n-dimensions as characterized by a set of numbers $\{x^i\}$, which serves as a representation only: to each point P there corresponds one and only one set $\{x^i\}$. There cannot be more than a finite number of points for which P can be represented by more than one set of $\{x^i\}$. (Example: the point of origin (0,0,0) in a Cartesian coordinate system is $(0, \theta, \varphi)$ in a polar coordinate system where θ and φ are arbitrary). The points P are therefore the important constituents of the structure of the space, and the coordinates only serve as a representation. They have no significance whatsoever to the interrelation among the points. The collection of <u>all</u> points of a space is often called a manifold.

(continued from p. 16)

$$ds^2 = \epsilon(\nu) dx_0^\nu dx_0^\nu$$

where $\epsilon(\nu) = \pm 1$. A Minkowskian space will be taken to be one such that $\epsilon(\nu) = -1$, $\nu = 1,2,3$, and $\epsilon(\nu) = +1$, $\nu = 4$. The set of $\epsilon(\nu)$ is the signature of the metric, and it can be shown to be invariant under coordinate transformations.

We shall assume the following property for the manifold:
(i) It has topological properties. That is, the concept of "being nearer to one point than the other" is applicable. This property enables us to specify the neighborhood to a point.
(ii) It is possible to establish a one-to-one correspondence relationship between the points in the neighborhood to points of an Euclidean space such that the concept of "being nearer to one point than the other" is the same. This property means that the space is locally Euclidean.
(iii) The one-to-one correspondence relationship is continuous in both spaces.
(iv) The space is simply connected. That is, the area of any closed curve canbe made to become zero by letting the length of the curve approach zero.

These properties are part of properties of an Euclidean space. Returning now to a four dimensional space, assume that the points of space are represented by a coordinate system $\{x^\nu\}$. Let there be a second coordinate system $\{x'^\nu\}$. Because of the one to one correspondence between a point and $\{x^\nu\}$ and $\{x'^\nu\}$ the x'^ν's can be expressed as functions of x^ν's:

$$x'^\nu = x'^\nu\left(\{x^\mu\}\right) \tag{14.12}$$

Similarly, we also have the inverse transformation:

$$x^{\mu} = x^{\mu}\left(\{x'^{\nu}\}\right) \tag{14.13}$$

Because of our requirement that coordinate systems are only representations of the points of a space, it is essential that inverse transformations exist for any coordinate transformation.

Consider the neighborhood to a point P There exists a one to correspondence between points in this neighborhood to points of a Euclidean space. We can define the distance between two points P and P' to be that between the corresponding points in the Euclidean space.* That is, we have

$$ds^2 = \epsilon(\nu)\, dx_0^{\nu} dx_0^{\nu} = \epsilon(\nu) \frac{\partial x_0^{\nu}}{\partial x^{\alpha}} \frac{\partial x_0^{\nu}}{\partial x^{\beta}} dx^{\alpha} dx^{\beta} = g_{\alpha\beta}\, dx^{\alpha} dx^{\beta} \tag{14.14}$$

$$g_{\alpha\beta} = \epsilon(\nu) \frac{\partial x_0^{\nu}}{\partial x^{\alpha}} \frac{\partial x_0^{\nu}}{\partial x^{\beta}}$$

where $\{x_o^{\nu}\}$ is the coordinate system for the corresponding Euclidean space. The coefficient $g_{\alpha\beta}$ is the metric tensor. In general $g_{\alpha\beta}$ are functions of $\{x^{\alpha}\}$ and $\{x^{\alpha}\}$ only. The metric tensor relates the distance between two neighboring points to the corresponding distances measured by the coordinates.

Since ds^2 describes an invariant relationship between P and P', in a different coordinate system $\{x'^{\nu}\}$ Eq. (14.14) becomes:

$$ds^2 = \epsilon(\nu) dx_0^{\nu} dx_0^{\nu} = \epsilon(\nu) \frac{\partial x_0^{\nu}}{\partial x'^{\alpha}} \frac{\partial x_0^{\nu}}{\partial x'^{\beta}} dx'^{\alpha} dx'^{\beta}$$
$$= g'_{\alpha\beta} \, dx'^{\alpha} dx'^{\beta} \tag{14.15}$$

The two expressions (14.14) and (14.15) are exactly equivalent. Using the relation

$$dx'^{\alpha} = \epsilon(\nu) \frac{\partial x'^{\alpha}}{\partial x^{\beta}} dx^{\beta} \tag{14.16}$$

we find that the coordinate transformation $\{x'^{\alpha}\} \to \{x^{\alpha}\}$ necessitates the following transformation law for $g_{\alpha\beta}$:

$$g_{\mu\nu} = g'_{\alpha\beta} \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} \tag{14.17}$$

Class of quantities which transform like (14.16) and (14.17) are known as tensors, which play a vital role in Riemannian Geometry.

(ii) Tensors.

(a) Definition and Algebra.

We define a contravariant tensor of rank n (in a four dimensional manifold as an entity with n^4 components denoted by n superscripted indices; the magnitude of the various components are generally functions of the coordinates used, but the various components of the tensor transform according to the following law:

$$\{x^{\nu}\} \rightarrow \{x'^{\nu}\} \tag{14.18}$$

$$T'^{\cdots\rho\sigma\delta\cdots} = T^{\cdots\alpha\beta\gamma\cdots} \cdots\cdots \frac{\partial x'^{\rho}}{\partial x^{\alpha}} \frac{\partial x'^{\sigma}}{\partial x^{\beta}} \frac{\partial x'^{\delta}}{\partial x^{\gamma}} \cdots\cdots$$

This is a generalization of Eq. (14.16) which is the law of transformation for an infinitesimal vector at a given point. A contravariant vector of rank 0 is a scalar whose magnitude is a function of the points of the manifold but is independent of the coordinates used.

We define a <u>covariant</u> <u>tensor</u> of rank n analogous to a contravariant tensor, by the following properties:

$$T'_{\cdots\rho\sigma\delta\cdots} = T_{\cdots\alpha\beta\gamma\cdots} \cdots\cdots \frac{\partial x^{\alpha}}{\partial x'^{\rho}} \frac{\partial x^{\beta}}{\partial x'^{\sigma}} \frac{\partial x^{\gamma}}{\partial x'^{\delta}} \cdots\cdots \tag{14.19}$$

Hence, if a tensor has zero components in one coordinate system it has zero components in <u>all</u> coordinate systems. We use subscripts to distinguish a covariant tensor from a contravariant tensor. It is seen that the metric tensor $g_{\mu\nu}$ is a covariant tensor of the second rank.

A mixed tensor has both superscripted and subscripted indices, the transformation law is a mixture of Eq. (14.18) and (14.19):

$$T'^{\cdots\rho\sigma\delta}_{\cdots\alpha\beta\gamma\cdots} = T^{\cdots\mu\nu\epsilon\cdots}_{\cdots\theta\eta\chi\cdots} \cdots\cdots \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \frac{\partial x'^{\delta}}{\partial x^{\epsilon}} \cdots \frac{\partial x^{\theta}}{\partial x'^{\alpha}} \frac{\partial x^{\eta}}{\partial x'^{\beta}} \frac{\partial x^{\chi}}{\partial x'^{\gamma}} \cdots \tag{14.20}$$

The law of addition of tensors applies only to tensors of the same rank and type (Contravariant or Covariant), and at the same point of the manifold. The rule is

$$T^{\cdots\rho\sigma\delta\cdots}_{\cdots\alpha\beta\gamma\cdots} + U^{\cdots\rho\sigma\delta\cdots}_{\cdots\alpha\beta\gamma\cdots} = (T+U)^{\cdots\rho\sigma\delta\cdots}_{\cdots\alpha\beta\gamma\cdots} \tag{14.21}$$

It is easily shown that $(T+U)^{\cdots\rho\sigma\delta\cdots}_{\cdots\alpha\beta\gamma\cdots}$ transforms like a tensor. The law of multiplication of tensors applies to any two tensors at the same point of the manifold. The resulting tensor has a rank equal to the sum of that of the two tensors. We have:

$$T^{\cdots\rho\sigma\delta\cdots}_{\cdots\alpha\beta\gamma\cdots} \cdot U^{\cdots\mu\nu\epsilon\cdots}_{\cdots\theta\eta\chi\cdots} = (TU)^{\cdots\rho\sigma\delta\cdots\mu\nu\epsilon\cdots}_{\cdots\alpha\beta\gamma\cdots\theta\eta\chi\cdots} \tag{14.22}$$

It is easily shown that $(TU)^{\cdots\rho\sigma\delta\cdots\mu\nu\epsilon\cdots}_{\cdots\alpha\beta\gamma\cdots\theta\eta\chi\cdots}$ transforms like a tensor. The simplest example of Eq. (14.22) is the tensor $A^{\mu\nu} = C^{\mu} D^{\nu}$ where C and D are vectors.

The rank of a tensor may be reduced by 2 by summing over all components of a pair of contravariant and covariant indices: For example

$$T^{\cdots\rho\sigma\delta}_{\cdots\rho\beta\gamma\cdots}$$

can be shown to transform like a tensor. This operation is called contraction. For example, the contraction of the tensor $A^{\mu} B_{\nu}$ is the scalar product $A^{\mu} B_{\nu}$ where A^{μ} and B_{ν} are two vectors.

There is no difference between contravariant and covariant tensors in rectangular Cartesian coordinates in a Euclidean space. The difference comes in when curvilinear coordinate systems are used, or when the space is curved.

The covariant component of a contravariant tensor $T^{\cdots\alpha\beta\gamma\cdots}$ is defined to be

$$T_{\cdots\alpha\beta\gamma\cdots} = T^{\cdots\mu\nu\epsilon\cdots}{}_{\cdots\cdots} g_{\alpha\mu} g_{\beta\nu} g_{\gamma\epsilon} \tag{14.23}$$

Thus the indices can be raised or lowered by contracting a new tensor made of products of the metric tensor and the contravariant tensor. In order to define the contravariant component for a covariant tensor $T_{\cdots\alpha\beta\gamma\cdots}$ we need the contravariant components of the metric tensor, which we define by the solution to the equation:

$$g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}_{\gamma} \tag{14.24}$$

It is easily shown that δ^{α}_{γ} transforms like a tensor. Eq. (14.24) is therefore invariant under coordinate transformations and

$$T^{\cdots\alpha\beta\gamma\cdots} = T_{\cdots\mu\nu\epsilon\cdots}{}^{\cdots\cdots} g^{\alpha\mu} g^{\beta\nu} g^{\gamma\epsilon} \cdots \tag{14.25}$$

A tensor is said to be symmetric or antisymmetric with respect to a pair of contravariant indices or covariant indices if the exchange of a pair of these two indices results in no change or only a change of sign:

$$T^{\cdots\alpha\cdots\beta\cdots} = \pm T^{\cdots\beta\cdots\alpha\cdots} \tag{14.26}$$

Any tensor can be written as a sum of symmetric and antisymmetric tensors. The metric tensor is, by its definition, a symmetric tensor. The symmetry properties of a tensor are preserved in tensor transformations.

Tensor densities of rank n and weight w (integer) are entities with n^4 components which transform according to the following law:

$$\mathfrak{J}'^{\cdots\mu\cdots} = \left|\frac{\partial x^{\alpha}}{\partial x'^{\beta}}\right|^{w} \cdots \frac{\partial x'^{\mu}}{\partial x^{\nu}} \cdots \mathfrak{J}^{\cdots\nu\cdots} \tag{14.27}$$

where $\left|\frac{\partial x^{\alpha}}{\partial x'^{\beta}}\right|$ is the Jacobian of the coordinate transformation. Similar laws of transformation exist for covariant quantities. The volume $\prod dx^{\nu}$ can be shown to be a tensor density of weight unity.

An important tensor density of weight one is the Levi-Civita tensor density $\epsilon^{\mu\nu\rho\sigma}$ defined as

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 \text{ when } \mu\nu\rho\sigma \text{ is an even permutation of } 1\ 2\ 3\ 4 \\ -1 \text{ when } \mu\nu\rho\sigma \text{ is an odd permutation of } 1\ 2\ 3\ 4 \\ \ \ 0 \text{ when any pair of } \mu\nu\rho\sigma \text{ are identical.} \end{cases} \tag{14.28}$$

Using $\epsilon^{\mu\nu\rho\nu}$ the determinant of a tensor of second rank $h_{\mu\nu}$ is given by

$$|h_{\mu\nu}| = \frac{1}{4}\epsilon^{\mu\alpha\rho\gamma}\epsilon^{\nu\beta\sigma\delta} h_{\mu\nu} h_{\alpha\beta} h_{\rho\sigma} h_{\gamma\delta} \tag{14.29}$$

b. Tensor Calculus.

We now obtain rules for differentiation for tensors. Ordinary derivatives of tensors are not invariant, since the coordinate intervals dx are not invariant. Further, the derivative of a function f is defined as follows:

$$\lim_{P' \to P} \left\{ \frac{f(\text{at point } P') - f(\text{at point } P)}{x^{\mu}(\text{at } P') - x^{\mu}(\text{at } P)} \right\} \tag{14.30}$$

As we have said, addition or subtraction of tensors can only be carried out at the same point. In orde to carry out the differentiation it is necessary to transport parallelly the value of the function at point P' to P, then take the difference and then let P' approach P.

Because tensors at different points of space transform differently, a transported tensor is different from the original vector by a small quantity. Consider the case of a vector A^{μ}

transported from a point P' to P:

$$\overline{A^{\mu}(P')} = A^{\mu}(P') - \delta A^{\mu} \tag{14.31}$$

where $\overline{A^{\mu}(P')}$ is the value of a vector $A^{\mu}(P')$ at P' transported to the point P. δA^{μ} generally depends on what coordinate systems we use. Its form is determined by requiring that $\overline{A^{\mu}(P')}$ transforms like a vector at P. Further $\delta A^{\mu} = 0$ when $A^{\mu} = 0$, and when $\Delta x^{\mu} = x^{\mu}(P') - x^{\mu}(P) = 0$. The simplest functional form of δA^{μ} for which these conditions are satisfied is:[1]

$$\delta A^{\mu} = -\Gamma^{\mu}_{\rho\sigma} A^{\rho} \Delta x^{\sigma} \tag{14.32}$$

where $\Gamma^{\mu}_{\rho\sigma}$ is some three-indexed quantity, which will be determined later. (It should be noted that $\Gamma^{\mu}_{\rho\sigma}$ is <u>not</u> a tensor, as will be shown later.) The derivative of A^{μ} with respect to x^{ν} is denoted by $A^{\mu}_{;\nu}$, is given by the equation:

$$A^{\mu}_{;\nu} = \lim_{P' \to P} \left\{ \frac{\overline{A^{\mu}(P')} - A^{\mu}(P)}{\Delta x^{\nu}} \right\} \tag{14.33}$$

From Eqs. (14.31) and (14.32) we find

$$A^{\mu}_{;\nu} = \frac{\partial A^{\mu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\rho\nu} A^{\rho} \tag{14.34}$$

---

[1]A space in which a law of parallel transport like Eq. (14.32) is said to have an affine connection.

We now determine $\Gamma^{\mu}_{\rho\nu}$. This we do with the aid of geodesics. Geodesics are equivalent to straight lines; an observer moving along a geodesic should always find himself moving in a direction tangent to the trajectory. This is one of the most important properties of a geodesic. By transforming $\Delta x^{\mu}(P) = x^{\mu}(P') - x^{\mu}(P)$ parallelly along itself from its origin P to its terminal point P', we obtain another vector $\Delta x^{\mu}(P')$ which extends from P' to P". We can again transform $\Delta x^{\mu}(P')$ parallelly along itself from P' to P", and so on. After many steps a broken curve will reach another point P_1. (Fig. 14.1). Letting $\Delta x^{\mu}(P) \to 0$ but keeping P and P' fixed, this broken curve will become a continuous curve. According to Eqs. (14.31) and (14. 32) we have the following relation between $\Delta x^{\mu}(P)$ and $\Delta x^{\mu}(P')$:

$$\Delta x^{\mu}(P) - \Gamma^{\mu}_{\rho\sigma}\Delta x^{\rho}\Delta x^{\sigma} = \Delta x^{\mu}(P') \tag{14.35}$$

Divide Eq. (14.34) by the square of the curve length between P and P', and in the limit P'$\to$ P, we have the equation for the curve

$$\lim_{\substack{P'\to P\\ ds\to 0}} \left\{ \frac{\frac{\Delta x^{\mu}(P')}{\Delta s} - \frac{\Delta x^{\mu}(P)}{\Delta s}}{\Delta s} + \Gamma^{\mu}_{\rho\sigma}\frac{\Delta x^{\rho}}{\Delta s}\frac{\Delta x^{\sigma}}{\Delta s} \right\} \tag{14.36}$$

$$= \frac{d^2x^{\mu}}{ds^2} + \Gamma^{\mu}_{\rho\sigma}\frac{dx^{\rho}}{ds}\frac{dx^{\sigma}}{ds} = 0$$

This curve will then have the property that the tangent to the curve at P (the direction is the same as Δx^{μ} at P) can be obtained by parallelly transportingthe tangent at another arbitrary point P_1 to P. Hence curves of this kind are geodesics. Compare Eq. (14.36) with Eq. (14.11) we find

$$\Gamma^{\mu}_{\rho\sigma} = \frac{1}{2} g^{\mu\nu} (g_{\nu\rho,\sigma} + g_{\sigma\nu,\rho} - g_{\rho\sigma,\nu})$$
$$= \left\{ {\mu \atop \rho\,\sigma} \right\} = \left\{ {\mu \atop \sigma\,\rho} \right\} = \Gamma^{\mu}_{\sigma\rho} \tag{14.37}$$

In a coordinate transformation $\Gamma^{\mu}_{\sigma\rho}$ does not transform like a tensor, the transformation law for $\Gamma^{\mu}_{\sigma\rho}$ is

$$\Gamma'^{\rho}_{\mu\nu} = \frac{\partial x'^{\rho}}{\partial x^{\sigma}} \frac{\partial x^{\gamma}}{\partial x'^{\mu}} \frac{\partial x^{\delta}}{\partial x'^{\nu}} \Gamma^{\sigma}_{\gamma\delta} + \frac{\partial x'^{\rho}}{\partial x^{\sigma}} \frac{\partial^2 x^{\sigma}}{\partial x'^{\mu} \partial x'^{\nu}} \tag{14.37a}$$

It is easily shown, though tedious, that, with the expression (14.37) for $\Gamma^{\mu}_{\sigma\rho}$, $A^{\mu}{}_{;\nu}$ transforms like a mixed tensor of the second rank in coordinate transformations. $A^{\mu}{}_{;\nu}$ is known as the covariant differentiation of A^{μ} with respect to the metric tensor $g_{\alpha\beta}$. By considering the parallel transport of two vectors A^{μ} and B^{ν} and by considering the parallel transport of their product $A^{\mu} B^{\nu}$ Eq. (14.37) is easily extended to contravariant tensors of arbitrary rank:

$$T^{\cdots\alpha\beta\gamma\cdots}{}_{;\nu} = T^{\cdots\alpha\beta\gamma\cdots}{}_{,\nu} + \cdots + \Gamma^{\alpha}_{\rho\nu} T^{\cdots\rho\beta\gamma\cdots} + \Gamma^{\beta}_{\rho\nu} T^{\cdots\alpha\beta\gamma\cdots} + \Gamma^{\gamma}_{\rho\nu} T^{\cdots\alpha\beta\gamma\cdots} + \cdots \tag{14.38}$$

We now obtain the covariant differentiation of $g_{\mu\nu}$ with respect to itself. First we note that at a particular point P the line element $ds^2 = g_{\alpha\beta} dx^\alpha dx^\gamma$ can be reduced to the form $ds^2 = dx'^\alpha dx'^\beta$ by shifting the origin to the point P and defining the new coordinates $\{x'^\mu\}$ by an equation similar to Eq. (14.35):

$$x^\mu = x'^\mu \Gamma^\mu_{\rho\sigma}(p) \; x'^\rho x'^\sigma \tag{14.39}$$

where $\Gamma^\mu_{\rho\sigma}(p)$ as a function of the old coordinates $\{x^\mu\}$ is evaluated at the point P. In the new coordinate system we find that $\Delta x'^\mu(P) = \Delta x^\mu(P')$, and so the covariant differentiation of a tensor is the same as the ordinary differentiation, i.e.

$$T^{\cdots\alpha\beta\gamma\cdots}{}_{;\nu} = \frac{\partial T^{\cdots\alpha\beta\gamma\cdots}}{\partial x'^\nu} = T^{\cdots\alpha\beta\gamma\cdots}{}_{,\nu} \tag{14.40}$$

Hence in the $\{x'^\mu\}$ system $g'_{\mu\nu,\sigma} = 0$. By differentiating the expression $g'^{\mu\nu} g_{\nu\eta} = \delta^\mu_\eta$ one further finds that $g'^{\mu\nu}{}_{,\sigma} = 0$. Since in the $\{x'^\mu\}$ system the covariant differentiation is the same as ordinary differentiation. Since a tensor vanishes identically if all its components vanish in one coordinate system we can conclude that

$$g^{\mu\nu}{}_{;\sigma} \equiv 0 \tag{14.41}$$

Similarly we can define a three indexed quantity $\gamma_{\rho\sigma}^{\ \ }$ such that the parallel transport of vector A_μ from P to P' results in a change which is

$$\delta A_\mu = \gamma_{\mu\sigma}^{\rho} A_\rho \Delta x^\sigma \tag{14.42}$$

and the covariant differentiation of a vector A_μ with respect to the metric tensor $g_{\mu\nu}$ is:

$$A_{\mu;\nu} = A_{\mu,\nu} - \gamma_{\mu\nu}^{\rho} A_\rho \tag{14.43}$$

It is easy to show that the following law holds for covariant differentiation:

$$\left(T^{\cdots\alpha\beta\cdots}_{\cdots\mu\nu\cdots}\right)_{;\epsilon} = \left(T^{\cdots\alpha\beta\cdots}_{\cdots\mu\nu\cdots}\right)_{,\epsilon} + \cdots + \Gamma^{\alpha}_{\rho\epsilon} T^{\cdots\rho\beta\cdots}_{\cdots\mu\nu\cdots} + \Gamma^{\beta}_{\rho\epsilon} T^{\cdots\alpha\rho}_{\cdots\mu\nu\cdots} + \cdots \gamma^{\rho}_{\mu\epsilon} T^{\cdots\alpha\beta\cdots}_{\cdots\rho\nu\cdots} \gamma^{\rho}_{\nu\epsilon} T^{\cdots\alpha\beta\cdots}_{\cdots\mu\rho\cdots} + \cdots \tag{14.44}$$

so that

$$\left(A^\mu B_\mu\right)_{;\alpha} = A^\mu_{;\alpha} B_\mu + A^\mu B_{\mu;\alpha} \tag{14.45}$$

$\gamma^{\alpha}_{\rho\sigma}$ can be obtained by the following manipulation:

$$\left(g^{\mu\nu}A_\nu\right)_{;\sigma} = g^{\mu\nu}A_{\nu;\sigma} = g^{\mu\nu}\left(A_{\nu,\sigma} - A_\alpha \gamma^{\alpha}_{\nu\sigma}\right) \tag{14.46}$$

$$\left(g^{\mu\nu}A_\nu\right)_{;\sigma} = \left(g^{\mu\nu}A_\nu\right)_{,\sigma} + \left(g^{\alpha\nu}A_\nu\right)\Gamma^{\mu}_{\alpha\sigma} \tag{14.47}$$

$$= g^{\mu\nu}A_{\nu,\sigma} + g^{\alpha\nu}A_\nu \Gamma^{\mu}_{\alpha\sigma} + g^{\mu\nu}_{\ \ ,\sigma}A_\nu$$

Multiply both equations by $g_{\beta\mu}$ and sum over μ , since $g_{\beta\mu} g^{\mu\nu} = \delta^{\nu}_{\beta}$ this results in eliminating $g^{\mu\nu}$ and replacing the index ν by β . Substitute for $\Gamma^{\mu}_{\alpha\sigma}$ Eq. (14.36) , one obtains the following equation:

$$A_{\alpha} \gamma^{\alpha}_{\beta\sigma} = A_{\epsilon} \Gamma^{\epsilon}_{\beta\sigma} \tag{14.48}$$

Hence (after replacing the dummy indices α and ϵ by δ)

$$\gamma^{\delta}_{\beta\sigma} = \Gamma^{\delta}_{\beta\sigma} \tag{14.49}$$

Hence the rule for covariant differentiation is

$$\begin{aligned}\left(T^{\cdots\alpha\cdots}_{\cdots\mu\cdots}\right)_{;\sigma} = \left(T^{\cdots\alpha\cdots}_{\cdots\mu\cdots}\right)_{,\sigma} + \cdots + \Gamma^{\alpha}_{\epsilon\sigma} T^{\cdots\epsilon\cdots}_{\cdots\mu\cdots} + \cdots \\ - \cdots - \Gamma^{\epsilon}_{\mu\sigma} T^{\cdots\alpha\cdots}_{\cdots\epsilon\cdots} - \cdots\end{aligned} \tag{14.50}$$

Using the above rule for differentiation and from the symmetry properties of $\Gamma^{\alpha}_{\beta\gamma}$ one can show that the following quantities are tensors:

(14.51)

(c) Curve Space and Curvature Tensor.

We have postulated that a curved space differs from an Eucleidean space by the important fact that no coordinate system $\{x_0^\nu\}$ exists such that the line element ds^2 may be reduced to the following form:

$$ds^2 = \epsilon(\nu) dx_0^\nu dx_0^\nu \tag{14.52}$$

at all points of space; in general the line element has the following form:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{14.53}$$

where $g_{\mu\nu}$ are functions of $\{x^\mu\}$. This property also manifests itself in that the parallel transport of a vector A^μ will introduce an affine connection:

$$\delta A^\mu = -\Gamma_{\rho\sigma}^{\cdot\,\mu} A^\rho \Delta x^\sigma \tag{14.54}$$

If the space is Euclidean, the affine connection vanishes. If a constant vector is displaced along a closed curve, and if the space is Eudlidean, then δA^μ integrated along this closed curve will vanish identically:

$$\oint \delta A^\mu = 0 \tag{14.55}$$

and the displaced vector coincides with the original vector. In such cases the space is said to be <u>integrable</u>, and parallelism can be established between two distant points, for, considering two distant points P and P_1, the parallel transport of a vector from P to P_1 will <u>not</u> depend on the path of choice. In general this is not true for a curved space; two vectors at two points on the longitude on the equator of the earth pointing towards north can be displaced parallelly along their meridian circles and when they reach the north pole they will intersect and the angle is the difference between their longitude.

We shall find conditions for which Eq. (14.55) is true. The integrability condition is the same as

$$A^{\mu}_{;\nu} = A^{\mu}_{,\nu} + \Gamma^{\mu}_{\alpha\nu} A^{\alpha} = 0 \tag{14.56}$$

We are to find a condition on $\Gamma^{\mu}_{\alpha\nu}$ such that Eq. (14.56) is true for arbitarary A^{μ}. Differentiating Eq. (14.56) with respect to x^{ω}, we have

$$A^{\mu}_{,} = -\Gamma^{\mu}_{\alpha\nu,\omega} A^{\alpha} - \Gamma^{\mu}_{\alpha\nu} A^{\alpha}_{,\omega} = -\left(\Gamma^{\mu}_{\alpha\nu,\omega} + \Gamma^{\mu}_{\alpha\nu} \Gamma^{\alpha}_{\rho\omega}\right) A^{\rho} \tag{14.57}$$

Exchange ν and ω, subtract the resulting equation from Eq. (14.57), we find the condition for $\Gamma^{\alpha}_{\beta\gamma}$;

$$\left(\Gamma^{\mu}_{\rho\nu,\omega} - \Gamma^{\mu}_{\rho\omega,\nu} - \Gamma^{\mu}_{\alpha\nu} \Gamma^{\alpha}_{\rho\omega} + \Gamma^{\mu}_{\alpha\omega} \Gamma^{\alpha}_{\rho\nu}\right) A^{\rho} = 0 \tag{14.58}$$

Define

$$R_{\nu\omega\rho}{}^{\mu} \equiv \Gamma^{\mu}_{\rho\nu,\omega} - \Gamma^{\mu}_{\rho\omega,\nu} - \Gamma^{\mu}_{\alpha\nu}\Gamma^{\alpha}_{\rho\omega} + \Gamma^{\mu}_{\alpha\omega}\Gamma^{\alpha}_{\rho\omega} \tag{14.59}$$

then the condition for Eq. (14.55) is

$$R_{\nu\omega\rho}{}^{\mu} = 0 \tag{14.60}$$

It can be shown by somewhat lengthy analysis that Eq. (14.60) is the necessary and sufficient condition for Eqs (14.55), and (14.56). $R_{\nu\omega\rho}{}^{\mu}$ is antisymmetric with respect to ν and ω. From Eq. (14.57) one obtains the expeession

$$A^{\mu}_{,\nu\omega} - A^{\mu}_{,\omega\nu} = A^{\mu}_{;\nu\omega} - A^{\mu}_{;\omega\nu} = R_{\nu\omega\rho}{}^{\mu} A^{\rho} \tag{14.61}$$

Since the left hand side transform like a tensor, the right hand side must transform like a tensor and hence, $R_{\nu\omega\rho}{}^{\mu}$ also transforms like a tensor. $R_{\nu\omega\rho}{}^{\mu}$ is called the <u>Riemann-Chistoffel curvature tensor</u>.

The Riemann-Chistoffel curvature tensor satisfies the following identities:

(i) $R_{\nu\omega\rho}{}^{\mu} = -R_{\omega\nu\rho}{}^{\mu}$ (antisymmetric with respect to ν and ω) (14.62)

(ii) $R_{\nu\omega\rho}{}^{\mu} + R_{\omega\rho\nu}{}^{\mu} + R_{\rho\nu\omega}{}^{\mu} = 0$ (the sum of permuting cyclically vanishes) (14.63)

(iii) The Bianchi Identity:

$$R_{\nu\omega\rho}{}^{\mu}{}_{;\sigma} + R_{\omega\sigma\rho}{}^{\mu}{}_{;\nu} + R_{\sigma\nu\rho}{}^{\mu}{}_{;\omega} = 0 \tag{14.64}$$

The Bianchi identity is obtained by first differentiating the covariant form of (14.61):

$$A_{\beta;\nu\omega} - A_{\beta;\omega\nu} = R_{\nu\omega\beta}{}^{\mu} A_{\mu} \tag{14.65}$$

then permuting the three indices $\beta\nu\omega$ cyclically, and adding the result, and then substituting the relation

$$A_{\alpha;\beta\gamma\delta} - A_{\alpha;\beta\delta\gamma} = -R_{\gamma\delta\alpha}{}^{\mu} A_{\mu;\beta} - R_{\gamma\delta\beta}{}^{\mu} A_{\beta;\mu} \tag{14.66}$$

Often one uses the covariant form of the curvature tensor which is

$$R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma}{}^{\mu} g_{\mu\delta} \tag{14.67}$$

Using the Chistoffel symbols of the first kind a similar expression to Eq. (14.60) may be obtained for $R_{\alpha\beta\gamma\delta}$

$$\begin{aligned} R_{\alpha\beta\gamma\delta} = {} & [\gamma\alpha, \delta]_{,\beta} - [\gamma\beta, \delta] \\ & + g^{\mu\nu}\left([\delta\alpha,\mu][\gamma\beta,\nu] - [\delta\beta,\nu][\gamma\alpha,\mu]\right) \end{aligned} \tag{14.68}$$

From the curvature tensor one can obtain other tensors by contraction. It is easily shown,by using the symmetry properties of the curvature tensor, that only four different types of contracted tensor exist, they are:

$$R^{\alpha}_{\alpha\beta\gamma},\ R^{\beta}_{\alpha\beta\gamma},\ R^{\delta}_{\alpha\beta\gamma}g^{\alpha\gamma},\ R^{\delta}_{\alpha\beta\gamma}g^{\beta\gamma} \tag{14.69}$$

and these four tensors are identical to each other apart from a sign. The most frequently used form of the contracted curvature tensor, designated by $R_{\mu\nu}$ is given by

$$R_{\mu\nu}=R^{\alpha}_{\alpha\mu\nu}=\left\{{\rho \atop \nu\rho}\right\}_{,\mu}-\left\{{\rho \atop \nu\mu}\right\}_{,\rho}-\left\{{\sigma \atop \rho\sigma}\right\}\left\{{\rho \atop \nu\mu}\right\}+\left\{{\rho \atop \sigma\mu}\right\}\left\{{\sigma \atop \nu\rho}\right\} \tag{14.70}$$

$R_{\mu\nu}$ is symmetric with respect to μ and ν. Because all terms are symmetric except $\left\{{\rho \atop \nu\rho}\right\}_{,\mu}$, to show this we note that

$$\begin{aligned}\left\{{\rho \atop \nu\rho}\right\}&=\tfrac{1}{2}g^{\sigma\rho}\left(g_{\sigma\nu,\rho}+g_{\sigma\rho,\nu}-g_{\nu\rho,\sigma}\right)\\ &=\tfrac{1}{2}g^{\sigma\rho}g_{\sigma\rho,\nu}=\tfrac{1}{2}\frac{g_{,\nu}}{g}=\left(\ln\sqrt{g}\right)_{,\nu}\end{aligned} \tag{14.71}$$

where $g = |g_{\alpha\beta}|$ is the determinant of g. Hence $\left\{{\rho \atop \nu\rho}\right\}_{,\mu}$ is symmetric with respect to μ and ν.

The contracted Bianchi identity is of interest as it ti directly related to the Einstein Field equation. Contracting Eq. (14.64), we obtain:

$$R_{\omega\rho;\sigma} + R_{\omega\sigma\rho}{}^{\mu}{}_{;\mu} + R_{\sigma\mu\rho}{}^{\mu}{}_{;\omega} = 0 \qquad (14.72)$$

Exchange μ and σ in the third term and use Eq. (14.62) we have

$$R_{\omega\rho;\sigma} + R_{\omega\sigma\rho}{}^{\mu}{}_{;\mu} + R_{\sigma\rho;\omega} = 0 \qquad (14.73)$$

Multiply Eq. (14.73) by $g^{\eta\rho}$ and sum over ρ, we find

$$R_{\omega}{}^{\eta}{}_{;\sigma} + R_{\omega\sigma}{}^{\eta\mu}{}_{;\mu} - R_{\sigma}{}^{\eta}{}_{;\omega} = 0 \qquad (14.74)$$

Use theproperty that $R_{\omega\sigma}{}^{\eta\mu} = R_{\sigma\omega}{}^{\mu\eta}$ and $R_{\sigma\omega}{}^{\mu\eta} = -R_{\omega\sigma}{}^{\mu\eta}$ further contraction of Eq. (14.74) with respect to η and ω results in the following equation:

$$R_{;\sigma} - 2R_{\sigma}{}^{\eta}{}_{;\eta} = 0 \qquad (14.75)$$

Multiply Eq. (14. 75) by $g^{\mu\sigma}$ and sum over σ, we obtain, finally:

$$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{;\nu} = 0 \qquad (14.76)$$

The expression in the parenthesis is often denoted by $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ The Einstein field equation is obtained b equating G to the stress energy tensor.

There are 16 components for $R_{\mu\nu}$ and $g_{\mu\nu}$, and 256 for $R_{\mu\nu\rho\sigma}$ the symmetry condition reduces the number of independent components to ten for $R_{\mu\nu}$ and $g_{\mu\nu}$, 20 for $R_{\mu\nu\rho\sigma}$.

Measurement Processes in General Relativity

The act of measuring the distances between points in space-time is of fundamental importance in the framework of general relativity. The usefulness of the concept of the existence of "curvature" in space-time depends on the existence of a means of measuring distance between neighboring points. Thus particular attention should be paid to defining the precise method to be used in making such measurements.

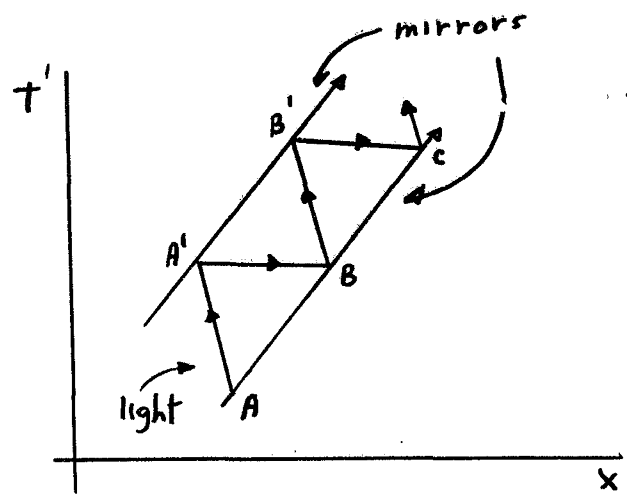
The classical techniques of comparing distances to standard "rods" and comparing times to standard "clocks" such as the period of the earth's orbit may not be useful in the general relativistic formalism. These standards depend ultimately on the constancy of the fundamental constants of nature, e, h, G and the like. Since, in certain formulations of the gravitational theory, these "constants' are expected to vary with the expansion of the universe, they are not useful in defining an absolute measuring apparatus.

What is needed is the establishment of a procedure for determining distances between space-time points which does not depend on these constants. Such a procedure has been developed by Marzke in his A.B. thesis at Princeton, and is outlined below.

Marzke's Method of Measurement

Marzke's method for the measurements of the proper times between events in space-time involves the use of a so-called "geodesic clock". The latter consists of two mirrors traveling along parallel paths in space-time with a light beam reflecting back and forth between them. The question of how one goes about constructing such a device will be discussed later. However, assuming this geodesic clock is set up, we can expect the paths of the light beam and mirrors to appear as follows.

Fig. 1



$x \equiv$ space coord.
$t' \equiv ct \equiv$ time coord.

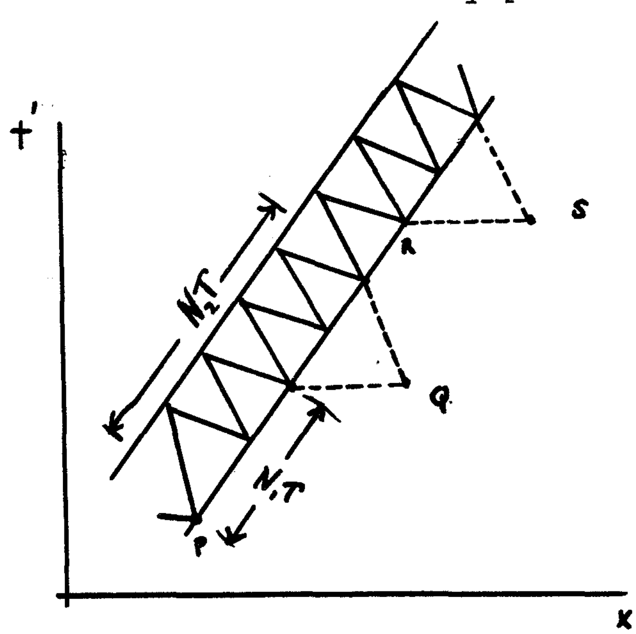
Since the paths A'B' and AC are parallel, the proper time intervals AB, BC, A'B', etc. are all equal and we can denote this interval by τ. If we focus our attention on one mirror, say that one traveling from A to B to C, etc., and record each intersection of the light with the mirror path, then we have a timing mechanism whose basic unit (of proper time) is τ .

It is to be especially noted that since each interval is marked by the intersection of a light beam and a material body (the mirror), it is independent of the Lorentz frame in which the event is observed, and of any of the possibly variable quantities like e, h, etc.

This apparatus can be used to measure the ratio of the proper time for two pairs of events. For example, suppose we wish to compare the proper time separating events P and Q with that separating R and S , we would proceed as follows.

First we choose one mirror which travels the space-time geodesic between events P and R, and then place the second mirror so that it traces out a nearby parallel to PR.

Fig. 2



At some point along PR we allow a light beam to be emitted from the mirror toward Q , and be reflected back (dotted line), and record the number of intervals our clock ticks from the happening of the event at P until the light beam is emitted ($N_1\tau$), and until the beam returns to the first mirror ($N_2\tau$). Then the distance PQ is given by

$$PQ = \sqrt{N_1 N_2}\,\tau$$

This can be seen if we transform to a system in which the mirrors are at rest. Then the situation is as diagrammed below

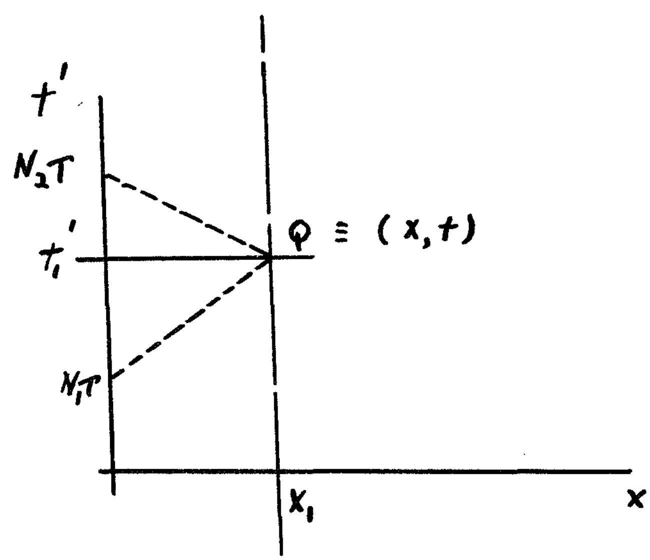


Fig. 3.

In this system, or any other, we have

$$(PQ)^2 = t^2 - x^2 = (t + x)(t - x)$$

Now $t_1' - N_1\tau$ is the time for the light beam to travel to x, which is just x (since we have defined $(t_1' - N_1\tau) \equiv c(t_1 - N_1\tau)$

hence

$$N_1\tau = t_1' - x$$

similarly,

$$N_2\tau = t_1' + x$$

and

$$PQ = \sqrt{N_1 N_2}\,\tau \quad .$$

If we carry out an identical procedure near the events R and we get

$$RS = \sqrt{N_3 N_4}\,\tau \quad .$$

Thus

$$\frac{PQ}{RS} = \sqrt{\frac{N_1 N_2}{N_3 N_4}}$$

and the intervals are compared in terms of only the absolute quantities N_1 N_2 N_3 and N_4 .

It remains to demonstrate a method for determining a parallel to a given world line. Once this is accomplished we could, in principle, choose a mirror traveling along that line and thus complete construction of a geodesic clock. Such a method was demonstrated by Marzke for a "locally flat" region of space-time. In principle we can always confine attention to a sufficiently small region such that curvature effects are smaller than a predetermined tolerance which we might set.

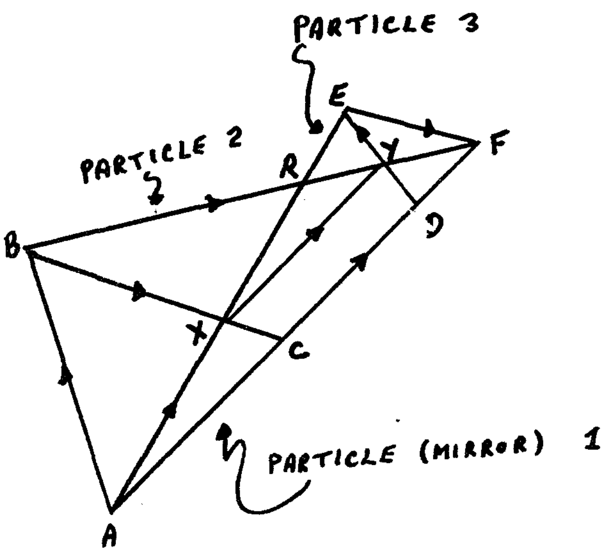


Fig. 4

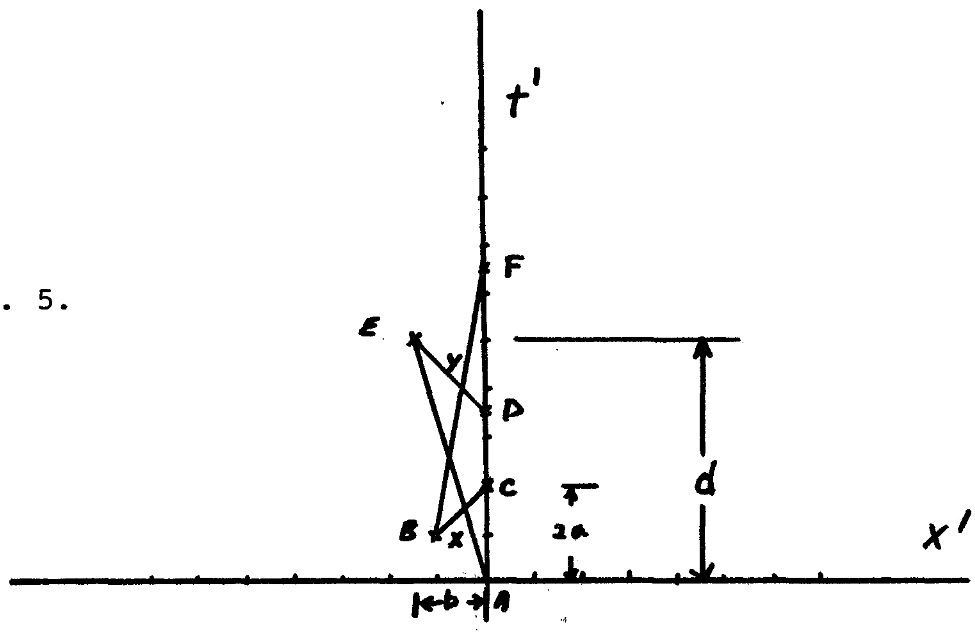
Thus we wish to construct a parallel to the world line ACDF of Figure 4. BF represents the path of a particle which intersects the particle traveling along ACDF, at F. AB and BF represents the path of a light ray emitted from particle 1, reflected from particle 2 and returning to intersect the path of particle 1 at C.

Now we consider a third particle whose path intersects particle 1 at A, the light ray BC at X, and BF, the path of particle 2 at R.

We imagine light rays constantly emitted from particle 1 and reflecting off particle 3 to return to the world line of particle 1. We select a particular light ray DEF,~~XY~~ which returns to particle 1 at F, and intersects particle 2 at Y.

A geometrical argument, based on Figure 5, in which we observe from a frame in which particle 1 is at rest, demonstrates that XY is the required parallel

Fig. 5.



Point A has been arbitrarily chosen to be at $(x'=0,\ t'=0)$, and C is assumed to correspond to the point 2a in our system. B then lies at $(-a,a)$ since ABC is a right $\measuredangle$ (path of a light ray). Point E lies at $(-b,d)$ thus defining the lengths b and d.

Since DEF is also a right angle, D and F must lie at $(o, d-b)$ and $(o, d+b)$ respectively.

Point X, the intersection of AE and BC then lies at

$$\left(\frac{-2ab}{b+d} \; , \; \frac{2ad}{b+d}\right)$$

While point y, the intersection of DE and BF lies at

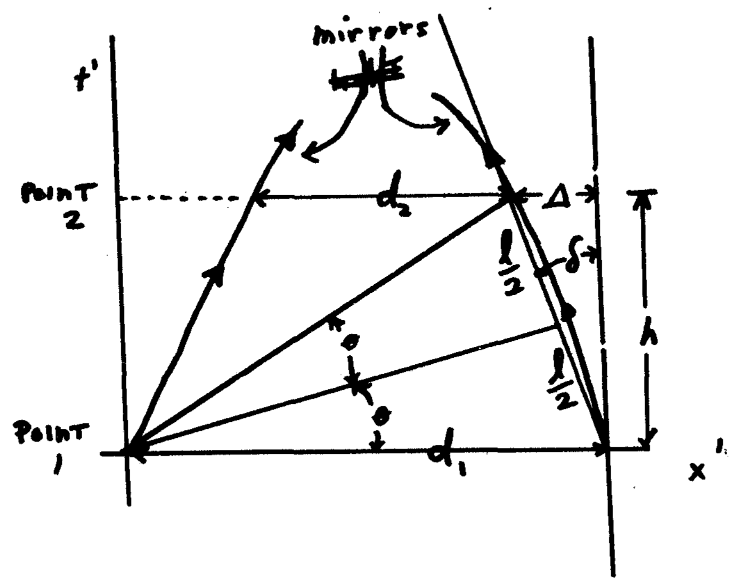
$$\left(\frac{-2ab}{b+d} \; , \; d-b, + \frac{2ab}{b+d}\right)$$

Thus the line XY is parallel to the t' axis and thus parallel to ACDF.

Extension to measurements in non "locally flat" regions.

The development of a measurement method in a locally flat region provides a procedure for making measurements in extended regions in which the curvature of space-time is non-negligible. Suppose, for example, our two mirrors begin motion on parallel world lines, the existence of the curvature of space time, or equivalently, their mutual attraction, results in the gradual decrease in separation between the mirrors. This results in variations in the basic unit of time, τ , and hence leads to inaccuracies in our geodesic clock. The magnitude of the inaccuracy is easily estimable. Suppose our mirrors are traveling the world line paths shown in Figure 6.

Fig. 6.



At any point along these curves, the time interval for a traversal of a light beam between the mirrors is proportional to the separation distance d. Then the fractional variation in this time for two points along the path is proportional to $\frac{d_2 - d_1}{d_1} = \frac{2\Delta}{d_1}$.

Assuming small curvatures we have

$$\sin\theta \sim \theta = \frac{l}{2d_1}$$

$$\sin\delta \sim \delta = \frac{\Delta}{l}$$

and, since $\delta = \theta$, then

$$\Delta = \frac{\ell^2}{2d_1}$$

$$\frac{\Delta\tau}{\tau} \sim \frac{2\Delta}{d_1} \sim \left(\frac{\ell^2}{d_1^2}\right) \sim \left(\frac{h^2}{d_1^2}\right)$$

Here, h is the elapsed time between the two pairs of points, and d_1 is roughly the radius of curvature of the world lines. For cases where h compares with d_1, this error can be a serious problem.

Suppose, however, that we divide the path h, into N portions, then the error in each portion is given by

$$\left(\frac{\Delta\tau}{\tau}\right)_p \sim \left(\frac{h}{Nd_1}\right)^2$$

assuming that a new geodesic clock is constructed in each portion. Then the overall error is then

$$\frac{\Delta\tau}{\tau} = N\left(\frac{\Delta\tau}{\tau}\right)_p \sim \frac{1}{N}\left(\frac{h}{d_1}\right)^2 \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty$$

Thus we can make our geodesic clock as precise as is necessary by using smaller and smaller intervals. Hence measurements of space time distances over regions of any size are feasible using a measuring technique which is independent of variations in the fundamental "constants" of the universe.

RIEMANN-CHRISTOFFEL TENSOR

The first covariant derivative of A^μ represents a tensor quantity

$$A^\mu{}_{;\nu} = \frac{\partial A}{\partial x^\nu} + \Gamma^\mu_{\alpha\nu} A^\alpha$$

The covariant derivatives of tensors of the second rank are defined analogously to covariant derivatives of vectors. For the case in which the tensor is $A^\mu{}_{;\nu}$ we have

$$A^\mu{}_{;\nu\omega} = \frac{\partial A^\mu{}_{;\nu}}{\partial x^\omega} + \Gamma^\alpha_{\nu\omega} A^\mu{}_{;\alpha} - \Gamma^\mu_{\alpha\omega} A^\alpha{}_{;\nu}$$

Substituting the expression for $A^\mu{}_{;\nu}$

$$A^\mu{}_{;\nu\omega} = \frac{\partial^2 A^\mu}{\partial x^\omega \partial x^\nu} + A^\alpha \Gamma^\mu_{\alpha\nu,\omega} + \Gamma^\mu_{\alpha\nu} A^\alpha{}_{,\omega} + \Gamma^\alpha_{\nu\omega} A^\mu{}_{,\alpha}$$

$$+ \Gamma^\alpha_{\nu\omega} \Gamma^\mu_{\varepsilon\alpha} A^\varepsilon - \Gamma^\mu_{\alpha\omega} A^\alpha{}_{,\nu} - \Gamma^\mu_{\alpha\omega} \Gamma^\alpha_{\varepsilon\nu} A^\varepsilon$$

The only terms that are altered if ν and ω are interchanged are the second and last terms. These two terms can be written as

$$A^\varepsilon \left(\Gamma^\mu_{\varepsilon\nu,\omega} - \Gamma^\mu_{\alpha\omega} \Gamma^\alpha_{\varepsilon\nu} \right)$$

The second covariant derivative of A^μ can also be written

$$A^\mu{}_{;\omega\nu} = A^\mu{}_{,\omega\nu} + \Gamma^\mu_{\alpha\omega} A^\alpha{}_{,\nu} + \Gamma^\alpha_{\omega\nu} A^\mu{}_{,\alpha} + \Gamma^\alpha_{\omega\nu} \Gamma^\mu_{\varepsilon\alpha} A^\varepsilon$$

$$- \Gamma^\mu_{\alpha\nu} A^\alpha{}_{,\omega} + A^\varepsilon \left(\Gamma^\mu_{\varepsilon\omega,\nu} - \Gamma^\mu_{\alpha\nu} \; \Gamma^\alpha_{\varepsilon\omega} \right)$$

Subtracting the two expressions for second covariant derivatives of A^μ one gets

$$A^\mu{}_{;\nu\omega} - A^\mu{}_{;\omega\nu} = \left(\Gamma^\mu_{\varepsilon\nu,\omega} - \Gamma^\mu_{\varepsilon\omega,\nu} + \Gamma^\mu_{\alpha\nu} \Gamma^\alpha_{\varepsilon\omega} - \Gamma^\mu_{\alpha\omega} \; \Gamma^\alpha_{\varepsilon\nu} \right) A^\varepsilon$$

$$= - R_{\nu\omega\varepsilon}{}^\mu A^\varepsilon$$

where $R_{\nu\omega\varepsilon}{}^\mu$ is the Riemann-Christoffel tensor.

From the definition of the Riemann-Christoffel tensor the following symmetries are apparent:

$$R_{\nu\omega\varepsilon}{}^\mu = - R_{\omega\nu\varepsilon}{}^\mu$$

$$R_{\nu\omega\varepsilon}{}^\mu + R_{\omega\varepsilon\nu}{}^\mu + R_{\varepsilon\nu\omega}{}^\mu = 0$$

Let is now consider a constant vector A^μ. The covariant derivative of the vector will vanish.

$$A^\mu{}_{;\nu} = A^\mu{}_{,\nu} + \Gamma^\mu_{\alpha\nu} A^\alpha = 0 \tag{1}$$

If this equation makes dA^{μ} a complete differential

$$\int dA^{\mu}$$

between any two points is independent of the path of integration. This vector can now be carried to any point by parallel displacement. Such a procedure will give a unique result independent of the path chosen.

The first task is thus to determine under what conditions, if any, dA^{μ} is a perfect differential.

From the theory of differential equations dA^{μ} will be a complete differential if $\Gamma^{\mu}_{\alpha\nu} A^{\alpha}$ is a complete differential. This condition is

$$-\frac{\partial}{\partial x^{\sigma}}\left(\Gamma^{\mu}_{\alpha\nu} A^{\alpha}\right) + \frac{\partial}{\partial x^{\nu}}\left(\Gamma^{\mu}_{\alpha\sigma} A^{\alpha}\right) = 0$$

$$-A^{\alpha}\left(\Gamma^{\mu}_{\alpha\nu,\sigma} - \Gamma^{\mu}_{\alpha\sigma,\nu}\right) - \Gamma^{\mu}_{\alpha\nu} A^{\alpha}{}_{,\sigma} + \Gamma^{\mu}_{\alpha\sigma} A^{\alpha}{}_{,\nu} = 0$$

Substituting from Eq. (1)

$$-A^{\alpha}\left(\Gamma^{\mu}_{\alpha\nu,\sigma} - \Gamma^{\mu}_{\alpha\sigma,\nu}\right) - \left[\Gamma^{\mu}_{\alpha\nu}\left(-\Gamma^{\alpha}_{\epsilon\sigma}\right) - \Gamma^{\mu}_{\alpha\sigma}\left(-\Gamma^{\alpha}_{\epsilon\nu}\right)\right] A^{\epsilon} = 0$$

$$\left(\Gamma^{\mu}_{\epsilon\sigma,\nu} - \Gamma^{\mu}_{\epsilon\nu,\sigma} + \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\alpha}_{\epsilon\nu} - \Gamma^{\mu}_{\alpha\nu}\Gamma^{\alpha}_{\epsilon\sigma}\right) A^{\epsilon} = 0$$

$$R_{\nu\sigma\epsilon}{}^{\mu} A^{\epsilon} = 0$$

The condition that dA^{μ} is a perfect differential is satisfied if the Riemann-Christoffel tensor vanishes. This means that if $R_{\nu\sigma\varepsilon}{}^{\mu} = 0$ we can construct an uniform vector field over the entire space.

This interesting property of the Riemann-Christoffel tensor is very intimately related to the properties of the space. For instance it is easy to imagine describing a uniform direction on a plane but there seems to be no analogy on a sphere. This would seem to indicate $R_{\nu\sigma\varepsilon}{}^{\mu}$ will vanish on a plane but not on a sphere.

Condition for Flat Space-Time

A definition of flat space-time is a region of the world where $g_{\mu\nu}$ are constants in some coordinate system.

When the $g_{\mu\nu}$ are constants the 3 index Christoffel symbols all vanish since they contain only derivatives of $g_{\mu\nu}$. However, since these symbols do not form a tensor they will not in general continue to vanish as other coordinate systems are considered in the same flat region.

Recall now that the Riemann-Christoffel tensor is composed of derivatives of $g_{\mu\nu}$ and products of derivatives. Obviously if $g_{\mu\nu}$ are constant the Riemann-Christoffel tensor will also vanish. And since it is a tensor it will continue to vanish as other coordinate systems are substituted in the same space.

Hence the vanishing of the Riemann-Christoffel tensor is a necessary condition for flat space time.

Another way to show thatspace is flat is to consider the angle between a unit vector and a geodesic.

$$\cos\Theta = g_{\alpha\beta}\lambda^{\alpha}\frac{dx^{\beta}}{ds} \quad \text{where } \lambda^{\alpha} = \text{unit vector}$$

Θ is an invariant angle if $\left(g_{\alpha\beta}\lambda^{\alpha}\frac{dx^{\beta}}{ds}\right)_{;\delta}\frac{dx^{\delta}}{ds} = 0$

$$g_{\alpha\beta;\delta} = 0$$

$$(\cos\Theta)_{;\delta}\frac{dx^{\delta}}{ds} = g_{\alpha\beta}\frac{dx^{\delta}}{ds}\left[\lambda^{\alpha}{}_{;\delta}\frac{dx}{ds} + \lambda^{\alpha}\left(\frac{dx^{\beta}}{ds}\right)_{;\delta}\right] \quad . \quad (2)$$

The second term is zero because $\frac{dx^{\delta}}{ds}\left(\frac{dx^{\beta}}{ds}\right)_{;\delta} = 0$ from the equation for a geodesic

$$\frac{dx^{\delta}}{ds}\frac{dx^{\beta}}{dx^{\delta}\,ds} + \frac{dx^{\delta}}{ds}\Gamma^{\beta}_{\alpha\delta}\frac{dx^{\alpha}}{ds} \Big/ \frac{d^2x}{ds^2} + \Gamma^{\beta}_{\alpha\delta}\frac{dx^{\alpha}}{ds}\frac{dx^{\delta}}{ds} = 0$$

Now consider $g_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta} = \text{const.}$

$$0 = \left(g_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta}\right)_{;\delta} = g_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta}{}_{;\delta}\frac{dx^{\delta}}{ds}$$

Since $g_{\alpha\beta}\frac{dx^{\delta}}{ds}$ and λ^{α} are in general not zero must be zero. Hence the first term in Eq. (2) is also zero. Thus $\cos\Theta$ is an invariant in flat space-time.

From the definition of the curvature tensor

$$R_{\nu\omega\rho}{}^{\mu} = \Gamma^{\mu}_{\rho\nu,\omega} - \Gamma^{\mu}_{\rho\omega,\nu} + \Gamma^{\mu}_{\alpha\omega}\Gamma^{\alpha}_{\rho\nu} - \Gamma^{\mu}_{\alpha\nu}\Gamma^{\alpha}_{\rho\omega}$$

one can straightforwardly verify the Bianchi Identity

$$R_{\nu\omega\rho}{}^{\mu}{}_{;\sigma} + R_{\omega\sigma\rho}{}^{\mu}{}_{;\nu} + R_{\sigma\nu\rho}{}^{\mu}{}_{;\omega} = 0$$

The proof is long and reasonably tedious. One must first determine the covariant derivative for a rank four tensor - then simply sum up the terms.

$$\begin{aligned}\Gamma^{\rho}_{\nu\rho} &= \frac{1}{2} g^{\sigma\rho}\left(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}\right) \\ &= \frac{1}{2} g^{\sigma\rho} g_{\sigma\rho,\nu}\end{aligned} \qquad (\ )$$

since the dummy indices σ and ρ can be interchanged.

Let g represent the determinant formed from the metric tensor

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{vmatrix}$$

dg is found by taking the differential of each $g_{\mu\nu}$ and multiplying by its co-factor $g.g^{\mu\nu}$. This formula is quite transparent if the determinant is expanded by cofactors and each term is differentiated. Then

$$\frac{dg}{g} = g^{\mu\nu} dg_{\mu\nu} = -g_{\mu\nu} dg^{\mu\nu}$$

Equation (2) now becomes

$$\Gamma_{\nu\rho}^{\rho} = \frac{1}{2g}\frac{\partial g}{\partial x^{\nu}} = \frac{\partial}{\partial x^{\nu}}\left(\ln\sqrt{-g}\right)$$

The minus sign occurs in the radical because it is always negative for real coordinates. Differentiating again

$$\Gamma_{\nu\rho,\mu}^{\rho} = \left(\ln\sqrt{-g}\right)_{,\nu\mu}$$

Curvature Relation for Empty Space

Contract the Bianchi identity $\nu = \mu$

$$R_{\nu\omega\rho}{}^{\mu}{}_{;\sigma} + R_{\omega\sigma\rho}{}^{\mu}{}_{;\nu} + R_{\sigma\nu\rho}{}^{\mu}{}_{;\omega} = 0$$

$$R_{\omega\rho;\sigma} + R_{\omega\sigma\rho}{}^{\mu}{}_{;\mu} - R_{\sigma\rho;\omega} = 0$$

Multiply by $g^{\eta\rho}$

$$R_{\omega}{}^{\eta}{}_{;\sigma} + R_{\omega\sigma}{}^{\eta\mu}{}_{;\mu} - R_{\sigma}{}^{\eta}{}_{;\omega} = 0$$

Contract over η and ω ($\eta = \omega$). Recall $R_{\omega\sigma}{}^{\eta\mu} = R_{\sigma\omega}{}^{\mu\eta}$ and $R_{\sigma\omega}{}^{\mu\eta} = - R_{\omega\sigma}{}^{\mu\eta}$

$$R_{;\sigma} - \underset{\substack{\parallel \\ R_{\sigma}{}^{\eta}{}_{;\eta}}}{R_{\omega\sigma}{}^{\mu\eta}{}_{;\mu}} - R_{\sigma}{}^{\eta}{}_{;\eta} = 0$$

$$R_{;\sigma} - 2 R_{\sigma}{}^{\eta}{}_{;\eta} = 0$$

Multiply by $g^{\mu\sigma}$

$$g^{\mu\sigma}\left(R_{;\sigma} - 2 R_{\sigma}{}^{\eta}{}_{;\eta}\right) = 0$$

$$\frac{1}{2} g^{\mu\sigma} R_{;\sigma} - R^{\mu\sigma}{}_{;\sigma} = -\left(R^{\mu\sigma} - \frac{1}{2} g^{\mu\sigma} R\right)_{;\sigma} = 0$$

This is the relation that R satisfies in empty space. Our result is very similar to Einstein's Equation

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{8\pi G}{c^2} T^{\mu\nu}$$

where $T^{\mu\nu}$ is the energy momentum.

<u>Special Case of Geodesic Equation</u>

The general form of the geodesic equation

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^{\alpha}_{\beta\gamma} \frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0$$

is rather intractable because 16 terms in β and γ must be summed for even one component of x^{α}. Let us consider the special case of a slowly moving particle in a space which is nearly flat ($g_{ii} \approx -1$; $g_{oo} \approx +1$; $g_{\alpha\beta} \approx 0$ for $\alpha \neq \beta$).

This is the classical situation and we expect Newton's gravitation law will hold.

$$\frac{d^2 x^{\mu}}{dt^2} = - \frac{GM}{r^3} x^{\mu} \qquad r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$$

Since the particle is moving slowly we can in general neglect the spatial portions of ds^2.

$$\frac{dx^o}{ds} \sim 1 \qquad\qquad \frac{dx^i}{ds} \backsim 1$$

since

$$ds^2 \approx c^2 \, dt^2$$

Then the geodesic equation becomes

$$\frac{d^2 x}{c^2 dt^2} + \Gamma^{\mu}_{oo} = 0$$

where

$$\Gamma^{\mu}_{oo} = \frac{1}{2} g^{\mu\beta} \left(\frac{\partial g_{\beta o}}{\partial x^o} + \frac{\partial g_{o\beta}}{\partial x^o} - \frac{\partial g_{oo}}{\partial x^{\beta}} \right)$$

Since $\frac{dx^o}{ds} \backsim 1$ $\Gamma^o_{oo} = 0$ and $\mu \neq 4$. Discarding off-diagonal terms of $g^{\mu\beta}$ and recalling $g^{\alpha\alpha} = -1$

$$\frac{d^2x^k}{c^2dt^2} = -\frac{1}{2}\frac{\partial g_{oo}}{\partial x^k}$$

Comparing this w Newton's Law of Gravitation -

$$\frac{d^2x^k}{c^2dt^2} = +\frac{1}{c^2}\frac{\partial}{\partial x^k}\text{(grav. potential)}$$

The gravitational potential ϕ_G is $-\frac{GM}{r^2}x^i$ so

$$\frac{1}{2}\frac{\partial}{\partial x^k}g_{oo} = -\frac{1}{c^2}\frac{\partial}{\partial x^k}\frac{GM}{r}$$

and

$$g_{oo} = -\frac{2GM}{c^2r}$$

However, g_{oo} is still arbitrary to within a constant. This constant is chosen so $g_{oo} \backsim 1$ when $r \rightarrow \infty$. Finally

$$g_{oo} = 1 - \frac{2GM}{c^2r}$$

Classical Field Theory

In the previous section we showed that the geodesic equation in the classical limit,

$$\bar{\nabla}^2 g_{oo} = \frac{8\pi G}{c^4} (\rho c^2)$$

with

$$g_{oo} = 1 - \frac{2\phi G}{c^2}$$

was equivalent to the Newtonian equation

$$\bar{\nabla}^2 \phi G = -4\pi G \rho_o$$

We must now generalize our idea of density in a way which is similar to the generalization of electric charge density to current density $\bar{J}^{\mu}$ in electrodynamics. The generalized density must be some kind of tensor since the geodesic equation contains the metric tensor.

The Energy-Momentum Tensor

If we define a Lagrangian density $\Lambda(q, \partial q/\partial x^{\alpha})$ such that the Lagrangian is given as

2

$$L = \int \Lambda \, dV ,$$

the equations of motion of the system are obtained from the principle of least action,

$$\delta S = \delta \int \Lambda (q, \partial q / \partial x^{\alpha}) \, dV \, dt . \\ = 0.$$

If we adopt the coordinates

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = ict$$

and write

$$ic \, dv \, dt = d\Omega$$

and

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 ,$$

then

$$S = -\frac{i}{c} \int \Lambda (q, \partial q / \partial x^{\alpha}) \, d\Omega .$$

Writing $\partial q / \partial x^{\alpha} = q_{,\alpha}$ we have

$$S = -\frac{i}{c} \int \left\{ \frac{\partial \Lambda}{\partial q} \delta q + \frac{\partial \Lambda}{\partial q_{,\alpha}} \delta q_{,\alpha} \right\} d\Omega .$$

This expression may be simplified using the identity,

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \Lambda}{\partial q_{,\alpha}}\,\delta q\right) = \delta q\,\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \Lambda}{\partial q_{,\alpha}}\right) + \frac{\partial \Lambda}{\partial q_{,\alpha}}\,\delta q_{,\alpha}\ ,$$

and the fact that δq vanishes at the end points of the integration. Now we have that

$$S = -\frac{i}{c}\int\left\{\frac{\partial \Lambda}{\partial q}\,\delta q \;-\; \delta q\,\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \Lambda}{\partial q_{,\alpha}}\right)\right\} d\Omega = 0 \quad ,$$

but since δq is arbitrary, this reduces to the equation of motion,

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \Lambda}{\partial q_{,\alpha}}\right) - \frac{\partial \Lambda}{\partial q} = 0 \quad .$$

We now look for some conserved quantities. We can write

$$\frac{\partial \Lambda}{\partial x^\alpha} = \frac{\partial \Lambda}{\partial q}\,\frac{\partial q}{\partial x^\alpha} + \frac{\partial \Lambda}{\partial q_{,\beta}}\,\frac{\partial q_{,\beta}}{\partial x^\alpha}$$

Using the equation of motion and the fact that

$$\frac{\partial q_{,\beta}}{\partial x^\alpha} = \frac{\partial^2 q}{\partial x^\alpha \partial x^\beta} = \frac{\partial q_{,\alpha}}{\partial x^\beta}$$

we have

$$\frac{\partial \Lambda}{\partial x^{\alpha}} = \frac{\partial}{\partial x^{\gamma}} \frac{\partial \Lambda}{\partial q_{,\gamma}} q_{,\alpha} + \frac{\partial \Lambda}{\partial q_{,\beta}} \frac{\partial q_{,\alpha}}{\partial x^{\beta}}$$

$$= \frac{\partial}{\partial x^{\beta}} \left(q_{,\alpha} \frac{\partial \Lambda}{\partial q_{,\beta}} \right) .$$

If we write $\frac{\partial \Lambda}{\partial x^{\alpha}}$ as $\frac{\partial \Lambda}{\partial x^{\beta}} \delta_{\alpha}^{\beta}$ we have

$$\frac{\partial}{\partial x^{\beta}} \left[\Lambda \delta_{\alpha}^{\beta} - q_{,\alpha} \frac{\partial \Lambda}{\partial q_{,\beta}} \right] = 0 .$$

Introducing the notation

$$T_{\alpha}^{\beta} = \Lambda \delta_{\alpha}^{\beta} - q_{,\alpha} \frac{\partial \Lambda}{\partial q_{,\beta}}$$

we have

$$\frac{\partial T_{\alpha}^{\beta}}{\partial x^{\beta}} = 0$$

Thus the T_{α}^{β}'s represent some kind of conserved quantities. The T_{α}^{4}'s will be constant in time and we expect that the quantities,

$$\int T_{\alpha}^{4} \, dV ,$$

will be associated with energy and momentum. Thus we write the four-momentum, P_{α}, as

$$P_\alpha = \text{constant} \int T_\alpha{}^4 \, dV \quad .$$

The constant may be determined by considering P_4 which in our notation should be $i\,E/c$. We have

$$P_4 = \text{constant} \int T_4{}^4 \, dV$$

and

$$T_4{}^4 = - \dot{q} \frac{\partial \Lambda}{\partial \dot{q}} + \Lambda \quad . \qquad \left(\dot{q} = \frac{\partial q}{\partial t} \right)$$

By comparing this with the usual formula relating the energy to the Lagrangian, we see that $- T_4{}^4$ must be considered as the energy density of the system. Therefore,

$$P_\alpha = - \frac{i}{c} \int T_\alpha{}^4 \, dV \quad .$$

We can now generalize this result to the case that the integration is over an arbitrary 3-dimensional hypersurface element dS_γ , and

$$P = - \frac{i}{c} \int T_\alpha{}^\gamma \, dS_\gamma \quad .$$

This follows from Gauss' theorem

$$\oint T_{\alpha}{}^{\gamma} \, dS_{\gamma} = \int \frac{\partial T_{\alpha}{}^{\gamma}}{\partial x^{\gamma}} \, d\Omega = 0 \quad .$$

$T_{\alpha}{}^{\beta}$ is called the energy-momentum tensor.

It is also possible to construct other conserved quantities when $T_{\alpha}{}^{\beta}$ is symmetric. We write

$$M^{\alpha}{}_{\beta} = \int (x^{\alpha} \, dP_{\beta} - x^{\beta} \, dP_{\alpha})$$

$$= - \frac{i}{c} \int \left(x^{\alpha} \, T_{\beta}{}^{\epsilon} - x^{\beta} \, T_{\alpha}{}^{\epsilon} \right) dS_{\epsilon} \quad .$$

If M is to be conserved we require that

$$\frac{\partial}{\partial x^{\epsilon}} \left(x^{\alpha} \, T_{\beta}{}^{\alpha} - x^{\beta} \, T_{\alpha}{}^{\epsilon} \right) = 0 \quad ,$$

using the fact that

$$\frac{\partial x^{\alpha}}{\partial x^{\epsilon}} = \delta_{\epsilon}^{\alpha} \, , \quad \frac{\partial x^{\beta}}{\partial x^{\epsilon}} = \delta_{\epsilon}^{\beta} \quad , \quad \frac{\partial T_{\beta}{}^{\epsilon}}{\partial x^{\epsilon}} = 0 \quad ,$$

this condition reduces to

$$T_{\beta}{}^{\alpha} - T_{\alpha}{}^{\beta} = 0$$

$M^{\alpha}{}_{\beta}$ is the four-angular momentum of the system and will be conserved only if $T_{\alpha}{}^{\beta}$ is a symmetric tensor.

$T_\alpha{}^\beta$ is in general not symmetric, but it can be made symmetric by adding to it a term of the form

$$\frac{\partial \phi_\alpha{}^{\beta\gamma}}{\partial x^\gamma}$$

with

$$\phi_\alpha{}^{\beta\gamma} = -\phi_\alpha{}^{\gamma\beta} .$$

Since

$$\frac{\partial \phi_\alpha{}^{\beta\gamma}}{\partial x^\beta \partial x^\gamma} = -\frac{\partial \phi_\alpha{}^{\gamma\beta}}{\partial x^\gamma \partial x^\beta} = \frac{\partial \phi_\alpha{}^{\gamma\beta}}{\partial x^\gamma \partial x^\beta} = 0,$$

$T_\alpha{}^\beta$ still satisfies the equation

$$\frac{\partial T_\alpha{}^\beta}{\partial x^\beta} = 0$$

Energy-Momentum Tensor for a System of Free Particles.

The mass density for this system can be written as

$$\rho = \sum_A m_A \, \delta(\bar{r} - \bar{r}_A)$$

Then the four-momentum density will be $\rho c u^\alpha$ with $U^\alpha = dx^\alpha/dS$. Since this density is equal to $-\frac{i T^{\alpha 4}}{c}$ we have $T^{\alpha 4} = i\rho c^2 U^\alpha$. But since the mass density is the time component of the four-vector $\frac{\rho}{i}\,\frac{dx^{\beta}}{ds}$, we expect that

$$T^{\alpha\beta} = \rho\, c^2\, U^{\alpha}\, U^{\beta}$$

In this notation $U^{\alpha} U^{\alpha} = -1$, therefore

$$T^{\alpha\alpha} = -\rho\, c^2$$

$$T^{\alpha\alpha} \leq 0$$

The Energy-Momentum Tensor for a Perfect Fluid

The conservation equation for the energy-momentum tensor

$$\frac{\partial T_{\alpha}{}^{\beta}}{\partial x^{\beta}} = 0$$

can also be written as

$$\frac{1}{ic}\,\frac{\partial T_4{}^4}{\partial t} + \frac{\partial T_4{}^i}{\partial x^i} = 0$$

$$\frac{1}{ic}\,\frac{\partial T_i{}^4}{\partial t} + \frac{\partial T_i{}^j}{\partial x^j} = 0$$

where i & j take on the values, 1, 2, 3 . If we integrate over a volume V in three spaces the first equation becomes

$$\frac{1}{ic}\,\frac{\partial}{\partial t}\int T_4{}^4\, dV + \int \frac{\partial T_4{}^i}{\partial x^i}\, dV$$

or by Gauss' Theorem,

$$\frac{\partial}{\partial t} \int_V (- T_4{}^4) \, dV = i c \oint_S T_4{}^i \, df_i$$

where the surface S surrounds the volume V .
The expression on the left is the rate of change of the energy contained in the volume V, and therefore the quantity on the right is the amount of energy transferred across S per unit time.

From the second equation we have

$$\frac{\partial}{\partial t} \int_V (- \frac{i}{c} T_i{}^4) \, dV = - \oint_S T_i{}^j \, df_j \quad .$$

On the left is the change of momentum of the system in the volume V per unit time; therefore $\oint T_i{}^j \, df_j$ is the momentum emerging from the volume V per unit time. The component $T_i{}^j$ of this tensor is the amount of the i^{TH} component of the momentum passing in unit time ~~through a unit time~~ through a unit surface perpendicular to the x^β axis.

Now, if we consider a perfect fluid, the energy-momentum tensor in the rest frame of a fluid element dV will be

$$T_i{}^j = p \, \delta_i{}^j$$

$$T_i{}^4 = 0$$

$$- T_4{}^4 = \rho c^2 = \epsilon$$

where P is the pressure and ρ is the density. We can write this as

$$T^{\alpha\beta} = \begin{vmatrix} P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & -\epsilon \end{vmatrix}$$

The requirement that $T^{\alpha\alpha} \leq 0$ leads to the condition $P \leq \epsilon/3$
In this notation we have been using

$$x_1 = x_1, \quad x_2 = y_1, \quad x_3 = z, \quad x_4 = i\,c\,t$$

and

$$ds^2 = c^2\,dt^2 - dx^2 - dy^2 - dz^2 \quad .$$

This requires a metric tensor of the form

$$g_{\mu\nu} = g^{\mu\nu} = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{vmatrix}$$

Thus we have that

$$T_\alpha{}^\beta = g_{\alpha\mu} T^{\mu\beta} = \begin{vmatrix} -P & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & \epsilon \end{vmatrix}$$

If we now change to the coordinates

$$x_1' = x, \quad x_2' = y, \quad x_3' = z, \quad x_4' = ct$$

we have that

$$T'_\nu{}^\mu = \frac{\partial x'^\mu}{\partial x^\beta} \frac{\partial x^\alpha}{\partial x'^\nu} T_\alpha{}^\beta = T_\nu{}^\mu$$

and

$$g'^{\mu\lambda} = g^{\alpha\beta} \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\lambda}{\partial x^\beta} = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

In these new coordinates we also have that

$$T^{\alpha\beta} = \begin{vmatrix} P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & -\epsilon \end{vmatrix}$$

We now label these proper coordinates (primed variables) with a subscript zero and consider a transformation to another arbitrary coordinate system.

$$T^{\mu\nu} = \frac{\partial x^{\mu}}{\partial x_o^{\alpha}} \frac{\partial x^{\nu}}{\partial x_o^{\beta}} T_o^{\alpha\beta}$$

$$= \frac{\partial x^{\mu}}{\partial x_o^{1}} \frac{\partial x^{\nu}}{\partial x_o^{1}} T_o^{11} + \frac{\partial x^{\mu}}{\partial x_o^{2}} \frac{\partial x^{\nu}}{\partial x_o^{2}} T_o^{22}$$

$$+ \frac{\partial x^{\mu}}{\partial x_o^{3}} \frac{\partial x^{\nu}}{\partial x_o^{3}} T_o^{33} + \frac{\partial x^{\mu}}{\partial x_o^{4}} \frac{\partial x^{\nu}}{\partial x_o^{4}} T_o^{44}$$

and

$$g^{\mu\nu} = \frac{\partial x^{\mu}}{\partial x_o^{\alpha}} \frac{\partial x^{\nu}}{\partial x_o^{\beta}} g_o^{\alpha\beta} .$$

If we add

$$\frac{\partial x}{\partial x_o^{4}} \frac{\partial x}{\partial x_o^{4}} P - \frac{\partial x}{\partial x_o^{4}} \frac{\partial x}{\partial x_o^{4}} P$$

to the equation for $T^{\mu\nu}$ above, we find that

$$T^{\mu\nu} = - g^{\mu\nu} P + (P + \epsilon) \frac{\partial x^{\mu}}{\partial x_o^{4}} \frac{\partial x^{\nu}}{\partial x_o^{4}} .$$

But because the x_o^{α} are proper coordinates,

$$dx_o^{\,4} = dS_o = dS$$

and

$$T^{\mu\nu} = -g^{\mu\nu} P + (P + \epsilon) \frac{\partial x^{\mu}}{dS} \frac{\partial x^{\nu}}{dS}$$

Field Equations

If we wish to obtain the Newtonian equations of motion in the weak field case, the field equation must have the form

$$\text{"}\square\text{"}\, g^{\alpha\beta} = (\text{constant}) \quad T^{\alpha\beta}$$

where "$\square$" is some second order operator. In the above discussion we have shown that

$$\frac{\partial T_{\alpha}{}^{\beta}}{\partial x^{\beta}} = 0$$

for a Lorentz metric. For the case of a general metric, this condition becomes

$$T^{\alpha\beta}{}_{;\beta} = 0$$

This leads to the requirement that

$$\left(\text{"}\Box\text{"} \, g^{\alpha\beta} \right)_{;\beta} = 0$$

The [redacted] SIMPLEST QUANTITY which will fulfill these conditions is

$$R^{\alpha\beta} - 1/2 \, g^{\alpha\beta} R \quad .$$

Thus we obtain the equations

$$R^{\alpha\beta} - 1/2 \, g^{\alpha\beta} R = \text{const.} \quad T^{\alpha\beta}$$

which are called Einstein's field equations.

SIMPLE SOLUTION OF FIELD EQUATION

We would now like to determine the solution of a particular problem - an isolated particle continually at rest at the origin. In obtaining this solution we shall be guided by our knowledge of the type of solution of such a particle. In particular -

1) at infinity Newtonian field theory should be applicable.

$$ds^2 = -(dx^2 + dy^2 + dz^2) + dt^2$$

2) the solution should be spherically symmetric

3) time is defined in units of 1/c - the length of time for light to travel 1 cm. $1/c \backsim 3.34 \times 10^{-11}$ sec.

   length is defined as usual in terms of cm.

   mass is defined as $c^2/G = 1.39 \times 10^{28}$ grams

The interval for Euclidean Space in terms of spherical polar coordinates is

$$ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + dt^2$$

The most general form of the interval possible without destroying the sperical symmetry in space is

$$ds^2 = -U(r)dr^2 - V(r)(r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + W(r)dt^2$$

Let $r_1^2 = V(r)r^2$. Then ds^2 becomes

$$ds = - U_1(r_1)dr_1^2\ r_1^2(d\theta^2 + \sin^2\theta\, d\phi^2) + W_1(r_1)dt^2$$

If the functions U_1 and W_1 differ from unity by only a small amount, both r and r_1 will have approximately the properties of the vector r in Euclidean space. Hence it makes no difference which we choose as our fundamental radius vector. Choose r_1 and drop the suffix.

$$ds^2 = - e^{\lambda}\ dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + e^{\nu}\ dt^2$$

where λ and ν are functions of r only.

From our previous specification that as $r \to \infty$ the gravitational field diminishes, λ and ν must go to zero as $r \to \infty$.

The metric tensor $g_{\mu\nu}$ becomes

$$g_{\mu\nu} = \begin{pmatrix} -e^{\lambda} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & e^{\nu} \end{pmatrix}$$

The determinant g obviously is simply $-e^{\lambda} e^{\nu} r^4 \sin^2\theta$.

Since

$$g_{\mu\nu} g^{\mu\nu} = 1 \qquad g^{\mu\nu} = \frac{1}{g_{\mu\nu}}\ .$$

$$g^{\mu\nu} = \begin{pmatrix} -e^{-\lambda} & 0 & 0 & 0 \\ 0 & -1/r^2 & 0 & 0 \\ 0 & 0 & -1/r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & e^{-\nu} \end{pmatrix}$$

The $g^{\mu\nu}$ vanish whenever $\mu \neq \nu$ so

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right) \quad \text{not summed}$$

The following cases are possible

$$\Gamma^{\mu}_{\mu\mu} = \frac{1}{2} g^{\mu\mu} \frac{\partial g_{\mu\mu}}{\partial x^{\mu}} = \frac{1}{2} \frac{\partial}{\partial x^{\mu}} (\ln g_{\mu\mu})$$

$$\Gamma^{\nu}_{\mu\mu} = - \frac{1}{2} g^{\nu\nu} \frac{\partial g_{\mu\mu}}{\partial x^{\nu}}$$

$$\Gamma^{\nu}_{\mu\nu} = \frac{1}{2} g^{\nu\nu} \frac{\partial g_{\nu\nu}}{\partial x^{\mu}} = \frac{1}{2} \frac{\partial}{\partial x^{\mu}} (\ln g_{\nu\nu})$$

$$\Gamma^{\sigma}_{\mu\nu} = 0$$

By carrying out the differentiation it is easy to determine all the nonvanishing Γ's.

$$\Gamma^{1}_{11} = \frac{1}{2} \qquad \Gamma^{3}_{23} = \cot\theta$$

$$\Gamma^{2}_{12} = \Gamma^{3}_{13} = 1/r \qquad \Gamma^{1}_{33} = -r\sin^2\theta\, e^{-\lambda}$$

$$\Gamma^{4}_{14} = \frac{1}{2}\nu' \qquad \Gamma^{2}_{33} = -\sin\theta\cos\theta$$

$$\Gamma^{1}_{22} = -re^{-\lambda} \qquad \Gamma^{1}_{44} = \frac{1}{2} e^{\nu-\lambda}\nu'$$

From the Einstein Eq:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

where

$$R_{\mu\nu} = R_{\alpha\mu\nu}{}^{\alpha} \qquad R_{\mu}{}^{\mu} = R$$

and

$$R_{\nu\omega\rho}{}^{\mu} = \Gamma^{\mu}_{\rho\nu,\omega} - \Gamma^{\mu}_{\rho\omega,\nu} - \Gamma^{\mu}_{\alpha\nu}\,\Gamma^{\alpha}_{\rho\omega} + \Gamma^{\mu}_{\alpha\omega}\,\Gamma^{\alpha}_{\rho\nu}$$

In empty space, however, the mass-energy tensor vanishes and the Einstein equation takes a much simpler form.

$$R = R_{\alpha}{}^{\alpha} = g^{\alpha\nu} R_{\alpha\nu}$$

Then

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\nu} R_{\alpha\nu} = 0$$

$$R_{\mu\nu} - \frac{1}{2} \delta_{\alpha\mu} R_{\alpha\nu} = 0$$

Finally in empty space we have

$$R_{\mu\nu} = 0$$

$\underline{\mu = \nu = 1}$

$$R_{\mu\nu} = \Gamma^{\alpha}_{1\alpha,1} - \Gamma^{\alpha}_{11,\alpha} + \Gamma^{\beta}_{1\alpha}\Gamma^{\alpha}_{1\beta} - \Gamma^{\alpha}_{11}\,\Gamma^{\mu}_{\alpha\mu} = 0$$

Substitute from the list of nonvanishing Γ's

$$R_{\mu\nu} = \cancel{\frac{\partial}{\partial r}\Gamma^1_{11}} + \frac{\partial}{\partial r}\Gamma^2_{12} + \frac{\partial}{\partial r}\Gamma^3_{13} + \frac{\partial}{\partial r}\Gamma^4_{14} - \cancel{\frac{\partial}{\partial r}\Gamma^1_{11}}$$

$$+ \cancel{\left(\Gamma^1_{11}\right)^2} + \left(\Gamma^2_{12}\right)^2 + \left(\Gamma^3_{13}\right)^2 + \left(\Gamma^4_{14}\right)^2 - \cancel{\Gamma^1_{11}\Gamma^1_{11}}$$

$$- \Gamma^1_{11}\Gamma^2_{12} - \Gamma^1_{11}\Gamma^3_{13} - \Gamma^1_{11}\Gamma^4_{14}$$

$$= -\frac{1}{r^2} - \frac{1}{r^2} + \frac{1}{2}\nu'' + \frac{1}{r^2} + \frac{1}{r^2} + \frac{1}{4}\nu'^2 - \frac{\lambda'}{2r} - \frac{\lambda'}{2r} - \frac{1}{4}\lambda'\nu'$$

$$= \frac{1}{2}\nu'' - \frac{1}{4}\nu'\lambda' - \frac{\lambda'}{r} + \frac{1}{4}\nu'^2 = 0$$

$\underline{\mu = \nu = 2}$

$$R_{\mu\nu} = \Gamma^\alpha_{2\alpha,2} - \Gamma^\alpha_{22,\alpha} + \Gamma^\beta_{\alpha 2}\Gamma^\alpha_{2\beta} - \Gamma^\beta_{\alpha\beta}\Gamma^\alpha_{22}$$

$$= \frac{\partial}{\partial\theta}\Gamma^3_{23} - \frac{\partial}{\partial r}\Gamma^1_{22} + \Gamma^1_{22}\Gamma^2_{21} + \Gamma^2_{12}\Gamma^1_{22} + \Gamma^3_{32}\Gamma^3_{23}$$

$$- \Gamma^1_{11}\Gamma^1_{22} - \Gamma^2_{12}\Gamma^1_{22} - \Gamma^3_{13}\Gamma^1_{22} - \Gamma^4_{14}\Gamma^1_{22}$$

$$R_{22} = -\csc^2\theta + e^{-\lambda}(1-\lambda' r) + \cot^2\theta + \frac{1}{2}r\lambda' e^{-\lambda} + \frac{1}{2}r\nu' e^{-\lambda}$$

$$= e^{-\lambda}\left[1 - \lambda' r + \frac{1}{2}r(\lambda' + \nu')\right] - 1 = 0$$

$\underline{\mu = \nu = 3}$

$$R_{33} = \Gamma^\alpha_{3\alpha,3} - \Gamma^\alpha_{33,\alpha} + \Gamma^\beta_{\alpha 3}\Gamma^\alpha_{3\beta} - \Gamma^\beta_{\alpha\beta}\Gamma^\alpha_{33}$$

$$= -\frac{\partial}{\partial r}\Gamma^1_{33} - \frac{\partial}{\partial\theta}\Gamma^2_{33} + \Gamma^1_{33}\Gamma^3_{31} + \Gamma^3_{13}\Gamma^1_{33} + \Gamma^2_{33}\Gamma^3_{32}$$

$$- \Gamma^1_{11}\Gamma^1_{33} - \Gamma^2_{12}\Gamma^1_{33} - \Gamma^3_{13}\Gamma^1_{33} - \Gamma^4_{14}\Gamma^1_{33}$$

$$= \sin^2\theta\, e^{-\lambda} - r\sin^2\theta\,\lambda' e^{-\lambda} + \cos^2\theta - \sin^2\theta - \sin^2\theta\, e^{-\lambda} - \cos^2\theta$$

$$+ \frac{1}{2} r \sin^2\theta\,\lambda' e^{-\lambda} + \sin^2\theta\, e^{-\lambda} + \frac{1}{2}\nu' r \sin^2\theta\, e^{-\lambda}$$

$$= \sin^2\theta\, e^{-\lambda}(1 - r\lambda' + \frac{1}{2} r\lambda' + \frac{1}{2} r\nu') - \sin^2\theta = 0$$

$$= \sin^2\theta\, e^{-\lambda}\left[1 + \frac{1}{2} r\,(\nu' - \lambda')\right] - \sin^2\theta = 0$$

$\mu = \nu = 4$

$$R_{44} = \Gamma^{\alpha}_{4\alpha,4} - \Gamma^{\alpha}_{44,\alpha} + \Gamma^{\beta}_{\alpha 4}\Gamma^{\alpha}_{4\beta} - \Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{44}$$

$$= -\frac{\partial}{\partial r}\Gamma^{1}_{44} + \Gamma^{4}_{14}\Gamma^{1}_{44} + \Gamma^{1}_{44}\Gamma^{4}_{41} - \Gamma^{1}_{11}\Gamma^{1}_{44}$$

$$- \Gamma^{2}_{12}\Gamma^{1}_{44} - \Gamma^{3}_{13}\Gamma^{1}_{44} - \Gamma^{4}_{14}\Gamma^{1}_{44}$$

$$= -\frac{1}{2}\nu'' e^{\nu-\lambda} + \frac{1}{2}\nu'(\lambda' - \nu')e^{\nu-\lambda} + \frac{1}{4}\nu'^2 e^{\nu-\lambda} - \frac{1}{4}\nu'\lambda' e^{\nu-\lambda}$$

$$- \frac{\nu' e^{\nu-\lambda}}{r}$$

$$R_{44} = e^{\nu-\lambda}\left[-\frac{1}{2}\nu'' - \frac{1}{4}\nu'^2 + \frac{1}{4}\nu'\lambda' - \frac{\nu'}{r}\right] = 0$$

The equations which must be satisfied are:

$$R_{11} = \frac{1}{2}\nu'' - \frac{1}{4}\nu'\lambda' - \frac{\lambda'}{r} + \frac{1}{4}\nu'^2 = 0 \qquad \text{(a)}$$

$$R_{22} = e^{-\lambda}\left[1 + \frac{1}{2} r(\nu' - \lambda')\right] - 1 = 0 \qquad \text{(b)}$$

$$R_{33} = \sin^2\theta\, e^{-\lambda}\left[1 + \frac{1}{2} r(\nu' - \lambda')\right] - \sin^2\theta = 0 \qquad \text{(c)}$$

$$R_{44} = e^{\nu-\lambda}\left[-\frac{1}{2}\nu'' - \frac{1}{4}\nu'^2 + \frac{1}{4}\nu'\lambda' - \frac{\nu'}{r}\right] = 0 \qquad \text{(d)}$$

In Eq. (d) the quantity within the brackets can be set equal to 0. Then adding Eq. (a) and Eq. (d)

$$\nu' + \lambda' = 0$$

Since ν and λ go to zero as $r \to \infty$ this quantity can be integrated to give

$$\nu = -\lambda$$

Eq. (b) becomes

$$e^{-\lambda}(1 + r\nu') = 1$$

$$e^{\nu}(1 + r\nu') = 1$$

Let $e^{\nu} = \gamma$. Then

$$\gamma + r\gamma' = 1$$

integrating

$$r\gamma = 1 - 2m$$

where 2 m is an integration constant. The integration constant is determined from previous comparison of the geodesic equation with Newton's Law of Gravitation.

$$\gamma = 1 - \frac{2m}{r} \quad .$$

Thus

$$e^{\nu} = 1 - \frac{2m}{r} \qquad e^{\lambda} = \frac{1}{1 - \frac{2m}{r}}$$

$$ds^2 = - \frac{dr^2}{1 - \frac{2m}{r}} - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 + (1 - \frac{2m}{r}) \, dt^2$$

As $r \to \infty$ this equation approaches the nonrelativistic expression for ds^2.

Note that for $r > 2m$ the interval ds^2 is the normal timelike interval. However for $r < 2m$ the interval becomes spacelike. At $r = 2m$ there exists a singularity - the Schwartzchild singularity.

At the Schwarzschild singularity $1 = \frac{2m}{r}$ in terms of cgs units

$$1 = \frac{2GM}{r\, c^2} = \frac{2GM^2}{r} \Big/ Mc^2$$

Therefore, at the "Schwarzschild radius" the gravitational self-energy of the body is equal to the rest energy. A particle of mass m would need $\frac{2GMm}{r}$ energy to escape so this body must convert <u>all</u> its self-energy in order to escape. A body initially within the Schwarzschild radius would never be able to escape to the "outside" world.

What is the "Schwarzschild radius" for some familiar bodies?

$$r_s = \frac{2GM}{c^2}$$

<u>protons</u>:

$$r_s = \frac{2 \times 6 \times 10^{-8} \times 1.7 \times 10^{-24}}{(3 \times 10^{10})^2} = 10^{-52} \text{ cm.}$$

<u>sun</u>:

$$r_s = 1.3 \times 10^{28} \times 2 \times 10^{33} = 2.6 \text{ km.}$$

If the total mass of the sun were contained within the "Schwartzschild radius" the density would be -

$$\rho = \frac{2 \times 10^{33}}{\frac{4}{3} \times 27 \times 10^{15}} \approx 10^{16} \text{ g/cc}$$

This gives the order of magnitude of density for which relativistic effects must be considered.

At the Schwarzschild radius the coefficient of dt^2 vanishes. This means that light leaving the body would be infinitely red shifted. An observer outside r_s would say the light takes an infinite time to reach him. All matter within r_s is thus invisible to outside observers. However, electrostatic or gravitational fields of a Schwartzschild singularity can be detected so the singularity can be felt but not seen.

The Schwarzschild singularity can be eliminated by the appropriate coordinate transformation. Let

$$r = \left(1 + \frac{m}{2\bar{r}}\right)^2 \bar{r} \quad .$$

The new metric becomes:

$$ds^2 = -\left(1 + \frac{m}{2\bar{r}}\right)^4 \left(d\bar{r}^2 + \bar{r}^2\, d\theta^2 + \bar{r}^2 \sin^2\theta\, d\phi^2\right) + \left(\frac{1-m/2\bar{r}}{1+m/2\bar{r}}\right) dt^2$$

This is called the isotropic polar coordinate system. It is the coordinate system on a falling body.

In isotropic rectangular coordinates the metric is:

$$ds^2 = -\left(1 + \frac{m}{2\bar{r}}\right)^4 (dx^2 + dy^2 + dz^2) + \frac{(1-m/2\bar{r})}{(1+m/2\bar{r})}\, dt^2$$

Aside from having no singularities these coordinates have some interesting advantages. To obtain the motion of a light pulse $ds = 0$ so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \frac{(1-m/2\bar{r})^2}{(1+m/2\bar{r})^6}$$

At a distance r from the origin the velocity of light is $\frac{1-m/2\bar{r}}{(1+m/2\bar{r})^3}$ in all directions.

Another consequence of the isotropic coordinates is that the coordinate length $(dx^2 + dy^2 + dz^2)^{\frac{1}{2}}$ of a small rigid rod does not change when the orientation of the rod is altered.

Static Field Equations for a Perfect Fluid

The non-vanishing components of contracted curvature tensor were found in the last section to be

$$R_{11} = \frac{\nu''}{2} - \frac{\lambda \nu'}{4} - \frac{\lambda'}{r} + \frac{\nu'^2}{2}$$

$$R_{22} = e^{-\lambda}\left[1 + \frac{r(\nu' - \lambda')}{2}\right] - 1$$

$$R_{33} = \sin^2\theta \; e^{-\lambda}\left[1 + \frac{r(\nu' - \lambda')}{2}\right] - \sin^2\theta$$

$$R_{44} = e^{\nu-\lambda}\left[-\frac{\nu''}{2} - \frac{\nu'^2}{4} + \frac{\nu'\lambda'}{4} - \frac{\nu'}{r}\right].$$

We can easily find the components of $R_\mu^{\ \nu}$ from

$$R_\mu^{\ \nu} = g^{\nu\alpha} R_{\mu\alpha} ,$$

and using the values of given in the previous section we find

$$R_1^{\ 1} = g^{1\alpha} R_{1\alpha} = g^{11} R^{11}$$

$$= e^{-\lambda}\left[\frac{\nu''}{2} - \frac{\nu'\lambda'}{4} - \frac{\lambda'}{r} + \frac{\nu'^2}{4}\right]$$

$$R_2^{\ 2} = -\frac{e^{-\lambda}}{r^2}\left[1 + \frac{r(\nu' - \lambda')}{2}\right] + \frac{1}{r^2}$$

$$R_3{}^3 = -\frac{e^{-\lambda}}{r^2}\left[1 + \frac{r(\nu' - \lambda')}{2}\right] + \frac{1}{r^2}$$

$$R_4{}^4 = e^{-\lambda}\left[-\frac{\nu''}{2} - \frac{\nu'^2}{4} + \frac{\lambda'\nu'}{4} - \frac{\nu'}{r}\right].$$

From these results we can calculate the scalar curvature R, and

$$R = R_\alpha^\alpha$$

$$= e^{-\lambda}\left[-\nu' - \frac{\nu'^2}{2} + \frac{\nu'\lambda'}{2}\right.$$

$$\left. - \frac{2}{r^2} - \frac{2(\nu' - \lambda')}{r}\right] + \frac{2}{r^2} .$$

We are now in a position to write out the field equations for a static perfect fluid. The field equations can be written as

$$R_\mu{}^\nu - \tfrac{1}{2} g_\mu{}^\nu R = -8\pi T_\mu{}^\nu ,$$

and for a perfect fluid

$$T^{\mu\nu} = (P + \rho)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - g^{\mu\nu} P$$

where P and ρ are the proper pressure and density of the fluid. Since we are dealing with the static case,

$$\frac{dr}{ds} = \frac{d\theta}{ds} = \frac{d\phi}{ds} = 0$$

$$\frac{dt}{ds} = e^{-\nu/2}$$

and

$$T^{\mu\nu} = (P + \rho)\, e^{-\nu}\, \delta^{\mu}_{4}\, \delta^{\nu}_{4} - g^{\mu\nu} P .$$

Using the fact that

$$T_{\mu}{}^{\nu} = g_{\mu\alpha}\, T^{\alpha\nu} ,$$

we obtain

$$T = \begin{matrix} -P & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -\rho \end{matrix}$$

Therefore we can write

$$-8\pi\, T_{1}{}^{1} = -e^{-\lambda}\left[\frac{\nu''}{2} - \frac{\nu'\lambda'}{4} - \frac{\lambda'}{r} + \frac{\nu'^{2}}{4}\right]$$

$$- \frac{e^{-\lambda}}{2} \left[- \nu'' - \frac{\nu'^2}{2} + \frac{\nu'\lambda'}{2} + \frac{2\lambda'}{r} - \frac{2\nu'}{r} \right.$$

$$\left. - \frac{2}{r^2} \right] - \frac{1}{r^2}$$

or

$$- 8\pi p = 8\pi T_1^{\ 1} = - e^{-\lambda} \left[\frac{\nu'}{r} + \frac{1}{r^2} \right] + \frac{1}{r^2}$$

Following the same procedure we find that

$$- 8\pi p = 8\pi T_2^{\ 2} = 8\pi T_3^{\ 3} = - e^{-\lambda} \left[\frac{\nu''}{2} \right.$$

$$\left. - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r} \right]$$

and

$$8\pi \rho = 8\pi T_4^{\ 4} = e^{-\lambda} \left[\frac{\lambda'}{r} - \frac{1}{r^2} \right] + \frac{1}{r^2}$$

Subtracting the second of these equations from the first and multiplying by $2/r$ we have

$$e^{-\lambda} \left[\frac{\nu''}{r} - \frac{\lambda'\nu'}{2r} + \frac{\nu'^2}{2r} + \frac{\nu' - \lambda'}{r^2} \right.$$

$$\left. - \frac{\nu'}{r^2} - \frac{2}{r^3} \right] + \frac{2}{r^3} = 0$$

and since we can write

$$8\pi \frac{dp}{dr} = e^{-\lambda}\left[\frac{\nu''}{r} - \frac{\nu'}{r^2} - \frac{2}{r^3}\right]$$

$$- e^{-\lambda}\lambda'\left[\frac{\nu'}{r} + \frac{1}{r^2}\right] + \frac{2}{r^3}$$

and

$$\frac{8\pi(p+\rho)}{2}\nu' = e^{-\lambda}\left[\frac{\lambda'}{r} + \frac{\nu'}{r}\right]\frac{\nu'}{2} ,$$

the above equation is equavalent to

$$\frac{dp}{dr} = -\frac{p+\rho}{2}\nu' .$$

Thus for a static and perfect fluid the field equations reduce to

$$8\pi p = e^{-\lambda}\left[\frac{\nu'}{r} + \frac{1}{r^2}\right] + \frac{1}{r^2}$$

$$8\pi \rho = e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{1}{r^2}\right] + \frac{1}{r^2}$$

$$\frac{dp}{dr} = -\frac{p+\rho}{2}\nu' .$$

The field equations in this form will be particularly useful in our further discussions.

PLANETARY ORBITS

The track of a particle moving freely in space-time is determined by the equations of a geodesic

$$\frac{d^2x^\sigma}{ds^2} + \Gamma^\sigma_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \quad .$$

Consider first $\sigma = 2$.

$$\frac{d^2x^2}{ds^2} + \Gamma^2_{12}\frac{dx^1}{ds}\frac{dx^2}{ds} + \Gamma^2_{21}\frac{dx^2}{ds}\frac{dx^1}{ds} + \Gamma^2_{33}\frac{dx^3}{ds}\frac{dx^3}{ds} = 0 \; .$$

From the previously defined Γ terms one can substitute and get

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0 \; .$$

Choose coordinates so the particle moves initially in the plane $\theta = \pi/2$. Then initially $\frac{d\theta}{ds} = 0$ and $\cos\theta = 0$ so $\frac{d^2\theta}{ds^2} = 0$. Thus the particle continues to move in this plane so $\theta = \pi/2$ can be substituted in the remaining equations. The equations for $\sigma = 1,3,4$ can be easily found by a similar method.

$$\frac{d^2r}{ds^2} + \frac{1}{2}\lambda'\left(\frac{dr}{ds}\right)^2 - r\,e^{-\lambda}\left(\frac{d\phi}{ds}\right)^2 + \frac{1}{2}e^{\nu-\lambda}\nu'\left(\frac{dt}{ds}\right)^2 = 0 \qquad (1)$$

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} = 0 \qquad (2)$$

$$\frac{d^2t}{ds^2} + \nu' \frac{dr}{ds}\frac{dt}{ds} = 0 \tag{3}$$

From Eq. (2) with $\frac{d\phi}{ds} = \alpha$

$$\frac{d\alpha}{ds} + \frac{2}{r}\frac{dr}{ds}\alpha = 0$$

$$\frac{d\alpha}{\alpha} = -\frac{2}{r}\,dr$$

$$\ln\alpha = -2\ln r + c$$

$$\alpha = \frac{1}{r^2}h \Rightarrow r^2\frac{d\phi}{ds} = h \quad \text{where } h \text{ is constant.}$$

This is the law of conservation of angular momentum for motion in a plane.

Multiply Eq. (3) by e^{ν}

$$e^{\nu}\frac{d^2t}{ds^2} + e^{\nu}\nu'\frac{dr}{ds}\frac{dt}{ds} = 0$$

$$\frac{d}{ds}\left(e^{\nu}\frac{dt}{ds}\right) = 0$$

$$\frac{dt}{ds} = K e^{-\nu} = \frac{K}{\gamma} \quad \text{where} \quad \text{is another constant of integration.}$$

Now use the metric equation, recalling that $\frac{d\theta}{ds} = 0$

$$1 = - e^{\lambda}\left(\frac{dr}{ds}\right)^2 - r^2 \sin^2 \theta \left(\frac{d\phi}{ds}\right)^2 + e^{\nu}\left(\frac{dt}{ds}\right)^2$$

Substituting the two integrals of the motion obtained above into the metric equation:

$$e^{\lambda}\left(\frac{dr}{ds}\right)^2 + r^2 \frac{h^2}{r^4} - e^{\nu} k^2 e^{-2\nu} = -1 \qquad \text{note } \gamma = e^{-\lambda} = e^{\nu}$$

But

$$\frac{d\phi}{ds} = h/r^2 \qquad \frac{dr}{ds} = \frac{h}{r^2}\frac{dr}{d\phi}$$

so

$$\frac{1}{\gamma}\left(\frac{h}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{h^2}{r^2} - \frac{k^2}{\gamma} = -1$$

Multiply through by $\gamma = 1 - \frac{2m}{r}$

$$\left(\frac{h}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{h^2}{r^2} = k^2 - 1 + \frac{2m}{r} + \frac{2m}{r}\frac{h^2}{r^2}$$

Let $1/r = U$

$$\left(\frac{dU}{d\phi}\right)^2 + U^2 = \frac{k^2 - 1}{h^2} + \frac{2m}{h^2} U + 2m\, U^3 \quad .$$

Differentiate with respect to Φ :

$$2\frac{dU}{d\Phi}\frac{d^2U}{d\Phi^2} + 2U\frac{dU}{d\Phi} = \frac{2m}{h^2}\frac{dU}{d\Phi} + 6\,m\,U^2\frac{dU}{d\Phi}$$

$$\frac{d^2U}{d\Phi^2} + U = \frac{m}{h^2} + 3m\,U^2$$

Compare this with the equations of a Newtonian orbit

$$\frac{d^2U}{d\Phi^2} + U = \frac{m}{h^2} \quad .$$

The ratio of $3mU^2$ to m/h^2 is $3h^2U^2$ or $3\left(r\frac{d\Phi}{ds}\right)^2$. For ordinary speeds this is an extremely small quantity. The effect of this correction on the shape of the orbit will be virtually undetectable.

ADVANCE OF PERIHELION

Although the small perturbation introduced into the orbital equation by relativity will not alter the shape of the orbit significantly, it can change the period of revolution. This change in the period is most easily observed as an advance of perihelion.

Neglecting the small relativistic perturbation the solution is

$$U = \frac{m}{h^2}\left[1 + e\cos(\Phi - \omega)\right]$$

where e is the eccentricity and ω is the longitude of perihelion.

Substitute this first U solution into the small term $3mU^2$. The equation becomes:

$$\frac{d^2U}{d\phi^2} + U = \frac{m}{h^2} + 3\frac{m^3}{h^4} + 6\frac{m^3}{h^4}\, e \cos(\phi - \omega) + \frac{3}{2}\,\frac{m^3}{h^4}\, e^2 \left[1 + \cos 2(\phi - \omega)\right]$$

The only additional term which can produce an observable effect is the term in $\cos(\phi - \omega)$; the period is right to produce a continually increasing effect by resonance.

What then is a particular solution of

$$U'' + U = A \cos \phi$$

Assume $U = a\phi \sin\phi + b\phi\cos\phi$ and substitute.

$$U' = a \sin\phi + a\phi\cos\phi + b\cos\phi - b\phi\sin\phi$$

$$U'' = a\cos\phi + a\cos\phi - a\phi\sin\phi - b\sin\phi - b\sin\phi - b\phi\cos\phi$$

$$U'' + U = 2a\cos\phi - 2b\sin\phi = A\cos\phi$$

Hence the particular solution of the relativistic orbital equation is

$$U_1 = \frac{3m^3}{h^4}\, e\phi \sin(\phi - \omega)$$

The complete solution is:

$$U = \frac{m}{h^2}\left[1 + e\cos(\Phi - \omega) + \frac{3m^2}{h^2} e\Phi\sin(\Phi - \omega)\right]$$

Assume this can be written

$$U = \frac{m}{h^2}\left[1 + e\cos(\Phi - \omega - \delta\omega)\right]$$

where $\delta\omega$ is very small and terms of order $(\delta\omega)^2$ can be neglected

$$\cos(\Phi - \omega - \delta\omega) = \cos(\Phi - \omega)\cos\delta\omega + \sin(\Phi - \omega)\sin\delta\omega$$

$$\approx \cos(\Phi - \omega) + \delta\omega\sin(\Phi - \omega) \quad .$$

Hence

$$\delta\omega = \frac{3m^2}{h^2}\Phi$$

While the planet moves through 1 revolution the perihelion advances a fraction of a revolution equal to

$$\frac{\delta\omega}{\Phi} = \frac{3m^2}{h^2} = \frac{3m}{a(1-e^2)}$$

from the equation of areas $h^2 = m\ell = ma(1-e^2)$.

Using Kepler's third law -

$$m = \left(\frac{2\pi}{T}\right)^2 a^3$$

$$\frac{\delta\omega}{\Phi} = \frac{12\pi^2 a^2}{c^2 T^2 (1-e^2)}$$

The advance of perihelion of Mercury is significant and the predicted value has been observed. However, one must be extremely careful in fixing the precise cause of the advance of perihelion. If the sun is sufficiently oblate the gravitational field will have quadruple terms which will cause advance of perihelion.

EXERCISE: Complete the advancement of perihelion of Mercury due to the effect of Jupiter.

mass of Jupiter $= 10^{-3} \odot$

dist. of Jupiter $= 4.5$ A.U.

FURTHER CONSEQUENCES OF THE SCHWARZSCHILD SOLUTION

Trajectory of a light ray.

Thus far we have not considered the question of how electromagnetic radiation travels in the Riemannian space. The nature of the trajectory of a light beam is not inherently contained in our theory as it is presently constructed. Additional postulates must be introduced analogously to the postulate that non-zero mass particles travel along geodesics.

It is natural to make the same assumption about the path. We postulate that light travels along geodesics in four dimensional space time. That is:

$$(d\tau)^2 = 0 \qquad . \qquad (1)$$

It becomes immediately apparent that other assumptions must be included if the nature of light propagation in the general relativistic formalism should bear any relationship to its nature in special relativity. In the case of the latter, we know that the constancy of the speed of light results in the vanishing of the proper time interval traversed by a light beam. That is

$$d\tau^2 = - c^2 \, dt^2 + dx^2 = - c^2 \, dt^2 + c^2 \, dt^2 = 0$$

i.e., We are on the light cone.

We thus are led to a second postulate about the path of a light beam

$$d\tau^2 = 0 \qquad (2)$$

(1) and (2) thus, confine light trajectories to the so-called "null geodesics", those paths whose proper time dependence on the four coordinates is an extremum; and in addition, have zero line dement along the path.

Our method of procedure is similar to that for material particles. That is we seek paths for which

$$\delta\int d\tau = \delta\int\left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{1/2} d\lambda = 0$$

Where λ is some parameter characterizing the path, in the case not the path length, since that is zero.

We are thus led to the equations of motion

$$\frac{d}{d\lambda}\left(\frac{dx^\alpha}{d\lambda}\right) + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0 \qquad (3)$$

A light ray in the Schwarzschild Field.

We will now consider the path of a light ray in the empty region of a space for which there is a spherically symmetric mass distribution at the origin. Solution of the field equations then yields the Schwarzschild metric.

$$d\tau^2 = -(1 - \frac{2m}{r})\, c^2\, dt^2 + (1 - \frac{2m}{r})^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \text{ where } m = \frac{GM}{c^2} \tag{4}$$

If we orient our axes so that the beam travels in the $z = 0$ plane, then $\theta = \pi/2$ and the calculation of the $\Gamma^{\alpha}_{\beta r}$ is identical to that demonstrated in the previous section on the trajectory of Mercury around the sun. The equations for the coordinates ϕ and t are also formally identical to those of that section:

$$r^2\dot{\phi} = h = \text{const.} \tag{5}$$

$$\left(1 - \frac{2m}{r}\right)\dot{t} = \ell = \text{const.} \tag{6}$$

Here, the dots denote differentiation with respect to λ. Rather than use the Euler-La Grange equation for r, we obtain a third independent equation by dividing the metric (4) by $d\lambda^2$ and setting $d\tau^2 = 0$

$$0 = -\left(1 - \frac{2m}{r}\right) c^2\, \dot{t}^2 + \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 + r^2\dot{\phi}^2$$

or

$$0 = -\frac{c^2\ell^2}{1 - \frac{2m}{r}} + \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 + \frac{h^2}{r^2}\ . \tag{7}$$

The solution of this equation can be achieved by making the substitution:

$$r = \frac{1}{u}$$

$$\dot{r} = -\frac{\dot{u}}{u^2} = -\frac{u'}{u^2}\dot{\phi} = -u' \; r^2 \dot{\phi} = -u' \; h$$

where

$$u' = \frac{du}{d\phi}$$

Then (7) becomes

$$0 = -\frac{c^2 \ell^2}{1-2mu} + \frac{u'^2 h^2}{1-2mu} + h^2 u^2$$

or

$$c^2 \ell^2 - h^2 u'^2 - h^2 u^2 (1 - 2mu) = 0 \qquad (8)$$

Differentiation of (8) with respect to ϕ , gives the equation for the trajectory :

$$- 2h^2 u'u'' - h^2 2uu'(1-2mu) + h^2 u^2 2mu' = 0$$

or

$$u'(u'' + u - 3mu^2) = 0 \qquad (9)$$

The solution $\left\{ \begin{array}{l} u' = 0 \\ u = \frac{1}{r} = \text{const.} \end{array} \right\}$ apparently does not correspond to anything physical, and is discarded in favor of solutions to the equation:

$$u'' + u - 3mu^2 = 0 \tag{10}$$

This equation for the trajectory of a light ray can be solved by a perturbation method when it is realized that the term in u^2 is much smaller than the term in u for typical value of the radial coordinate. That is, say for the sun,

$$\frac{3mu^2}{u} = \frac{3m}{r} = \frac{(3)GM}{c^2 r}$$

$$\approx \frac{(3)(7\times10^{-8})(2\times10^{33})}{10^{21}\ r} \sim \frac{5\times10^{5}\mathrm{cm}}{r}$$

$$= \frac{5\ \mathrm{km}}{r}$$

Thus, outside the surface of the sum the term $3mu^2$ can be treated as a perturbation. Then, writing $3m = \epsilon$, we look for a solution of (10) good to first powers of the parameter ϵ, of the form

$$U(\phi) = U_o + \epsilon v + O(\epsilon^2)$$

$$U'' = U_o'' + \epsilon v''$$

This trial solution in (10) gives

$$U_o'' + \epsilon v'' + U_o + \epsilon v = \epsilon\,(U_o + \epsilon v)^2 \quad . \tag{11}$$

Equating the zero order terms in ϵ gives

$$U_o'' + U_o = 0$$

or

$$U_o = A \cos(\phi + \delta) \tag{12}$$

or $U_o = A \cos\bar{\theta}$ after the substitution $\bar{\theta} = \phi + \delta$, thus rotating our axes about z .

This solution, good to zero order in ϵ , corresponds to

$$x = r \cos\theta = \frac{1}{A} = \text{const.}$$

That is, in this approximation the ray travels a straight line parallel to the y axis. Thus the effects of the sun's presence are completely neglected. ~~The e~~

The effect of the sun's perturbation is found by equating terms linear in ϵ , in (11)

$$v'' + v = U_o^2 = A^2 \cos^2\bar{\theta} \tag{13}$$

where now the differentiation refers to the variable $\bar{\theta}$.

The solution of (13) is easily found to be

$$v = \frac{2}{3} A^2 - \frac{1}{3} A^2 \cos^2\bar{\theta} \tag{14}$$

Combining (12) and (14) we have the trajectory, to first order in $\epsilon = 3m$

$$U = A \cos\bar{\theta} - mA^2 \cos^2\bar{\theta} + 2mA^2 \tag{15}$$

This solution corresponds to the light beam approaching the sun on a straight line, being deflected, and receding again along a path which asymptotically approaches another straight line.

The deflection of the ray is then measured by the angle between these asymptotes, which can be obtained by setting $r = \infty$ in (15).

$$A \cos\bar{\theta} - mA^2 \cos^2\bar{\theta} + 2mA^2 = 0 \tag{16}$$

(16) has the solution

$$\cos\bar{\theta} = \frac{3}{6mA}\left\{ 1 - (1 + \frac{8m^2A^2}{})^{\frac{1}{2}} \right\}. \tag{17}$$

The other solution to (16) is discarded since it yields $\cos\bar{\theta} > 1$.

To first order in the parameter $3m = \epsilon$, this becomes

$$\cos\bar{\theta} = \frac{3}{2\epsilon A}\left(1 - 1 + \frac{8}{18}\epsilon^2 A^2 \right) = \frac{3}{2A}\,\frac{4}{9}\,\epsilon\,\frac{A^2}{A}$$

$$\cos\bar{\theta} = 2mA \tag{18}$$

The significance of the parameter A is realized if $\bar{\theta}$ is set to zero in (15) then

$$U(0) = A + mA^2 \approx A$$

This is the point of closest approach of the light ray to the sun. That is

$$U(0) = \frac{1}{R} = A$$

where R is the distance at closest approach. Hence $\cos\bar{\theta} = \frac{2m}{R}$ corresponds to the asymptotes of the trajectory.

By our previous estimate of m, it is realized that for typical values of R, we have

$$\cos \bar{\theta} \approx 0$$

or $\bar{\theta} = \pm \frac{\pi}{2} \pm \delta$ where δ is small. Putting this into (18) gives

$$\sin \delta \approx \delta = \frac{2m}{R}$$

for either sign combination.

The overall deflection of a light beam, due to the presence of a spherically symmetric mass distribution at the origin is then

$$\Delta = 2\delta = \frac{4m}{R} \qquad (19)$$

As is well known, this result provides the basis for one of the three experimental checks of General Relativity.

For the case of a light beam which just grazes the rim of the sun, the deflection is

$$\Delta = \frac{4GM}{Rc^2} = 1.75''$$

Measurements of this quantity for starlight just passing the rim of the sun during an eclipse have yielded results which seem to confirm this prediction. Measurements made between 1919 and 1954 have fluctuated between 1.5" and 3" with a median near the somewhat high value of 2.2". A particularly accurate measurement made by Van Biesbroeck in 1952 yielded 1.7" ± .1", in excellent agreement with theory.

<u>Newtonian theory of the deflection of a particle by the sun</u>.

It is interesting to compare the above analysis to the Newtonian theory for the deflection of a particle by the sun. The particle mass does not enter these equations.

Suppose the trajectory is close to a straight line parallel to the y-axis and the point of closest approach is R.

Then the acceleration in the x-direction is

$$\ddot{X} = \frac{GMx}{r^3} = \frac{GMx}{(x^2+y^2)^{3/2}}$$

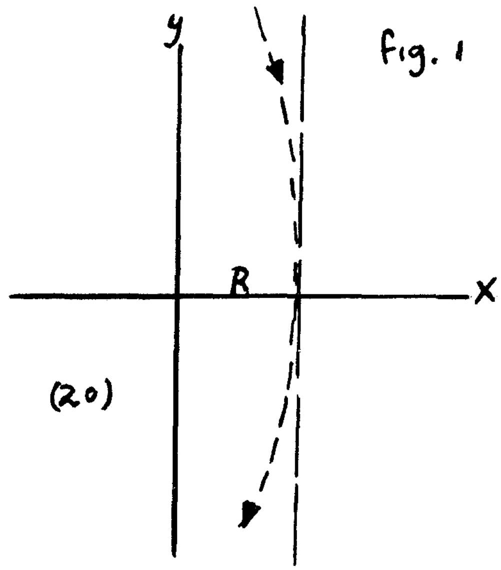


Fig. 1

or

$$x''c^2 = \frac{GMx}{(x^2+y^2)^{3/2}} \qquad (20)$$

In (20), x'' refers to differentiation with respect to y, and the particle is assumed to be travelling at a velocity very close to c, in which case

$$\dot{x} = \frac{dx}{dt} = x'\frac{dy}{dt} \approx x' c$$

$$\ddot{x} \approx c^2 x''$$

A further approximation is made in which we assume x is roughly equal to R for all y. Then

$$(x')' c^2 = \frac{GMR}{(y^2+R^2)^{3/2}}$$

Thus

$$\int_{x_0'}^{x'} dx' = \frac{GMR}{c^2}\int_0^y \frac{dy}{(y^2+R^2)^{3/2}}$$

If there is a true minimum of the radial coordinate at $y = 0$, then x_o is zero at that point. An integration gives

$$x' = \frac{GM}{c^2 R} \sin(\tan^{-1} \frac{y}{R})$$

$$x' = \frac{GM}{c^2 R} \frac{y}{(R^2 + y^2)^{1/2}} \approx \frac{GM}{c^2 R}$$

where the last step assumes we will only discuss the region $y^2 \gg R^2$, or $|y| \gg R$.

Another integration yields:

$$x(y) = \frac{GM}{c^2 R} y + R \tag{21}$$

where we have used the fact that $x(0) = R$. Equation (21) is the equation for the asymptotes to the trajectory at large values of y.

These asymptotes have a slope $\frac{GM}{c^2 R}$. Thus the total deflection is

$$\Delta = \frac{2GM}{c^2 R}$$

or exactly one-half that predicted by the General Relativity theory for the trajectory of a light ray.

A fourth test of General Relativity

Recently, Shapiro[1] has determined, using the Schwarzschild metric, the transit time for a radar beam to traverse the distance from the earth to one of the inner planets and return. He concludes that the difference in elapsed time between the cases when, first, the experiment is done when the earth and target planet are situated so that the beam passes close tothe sun, and second when the beam does not pass close to the sun; is large enough to be measured. Performing the experiment could provide a test of general relativity. To illustrate this, we calculate the transit time below.

It will be convenient to orient our axis as shown in Figure 2.

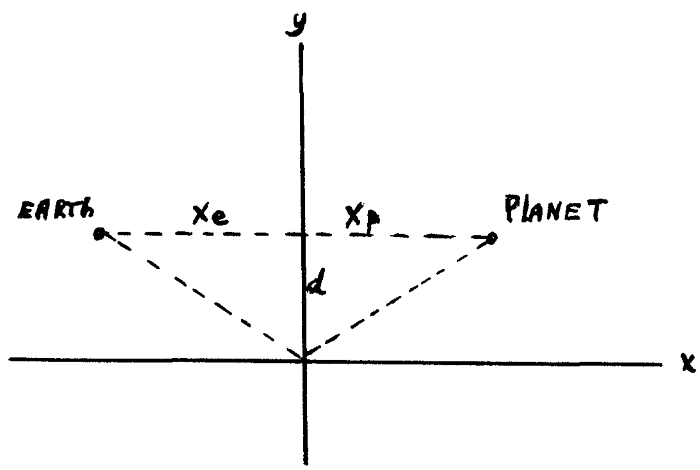


Fig. 2

[1] I. I. Shapiro, Phys. Rev. Ltr. 13, 26, 789-91, Dec. 1964.

Here we have considered the sun at the origin of coordinates and the earth and the inner planet to be separated by a distance $x_e + x_p$. In what follows, the space-time curvature due to the earth and planet will be neglected compared to the much larger influence of the sun. This allows us to use the Schwarzschild line element. We further neglect the small motions of the earth and planet.

A proper calculation at this point would involve first calculating the null geodesic trajectory between the transmitter and target, and then determining the transit time for a radar beam to travel this path and return. Such a calculation, however, which is quite tedious, is found to differ from the much simpler calculation for a radar beam which traverses a direct path to the planet, only by a term of the order of $m^2 = \left(\frac{GM}{c^2}\right) \sim (5 \text{ km})^2$. The length m is a small quantity compared to the distances in this problem.

We will perform the simpler calculation here, which will be accurate to terms of the order of m and includes effects due to the warping of space time by the sun, but not of the path deflection of the light beam.

The Schwarzschild metric, in spherical polar coordinates is written as

$$dT^2 = -(1 - \frac{2m}{r})\, c^2 dt^2 + (1 - \frac{2m}{r})^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \quad (2\text{:}$$

In terms of the cartesian coordinates:

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\phi$$

The Schwarzschild metric assumes the form

$$dT^2 = -(1 - \frac{2m}{r})\ c^2 dt^2 + \left(\delta_{ij} + \frac{2mx_i x_j}{r^3}(1 - \frac{2m}{r})^{-1}\right)\ dx^i dx^j \tag{24}$$

Now, in the (x,y) plane, with y = d = const. as in Figure (2) we have $dz = dy = 0$, and (24) considerably simplifies to

$$dT^2 = -(1 - \frac{2m}{r})\ c^2 dt^2 + \left(1 - \frac{2mx^2}{r^3}(1 - \frac{2m}{r})^{-1}\right)\ dx^2$$

Furthermore, the radar beam travels the null geodesic, $dT^2 = 0$. Hence we obtain

$$dt^2 = \left\{\left(1 - \frac{2m}{r}\right)^{-1} + \frac{2mx^2}{r^3}\left(1 - \frac{2m}{r}\right)^{-2}\right\} \frac{dx^2}{c^2}$$

$$\approx \left\{1 + \frac{2m}{r} + \frac{2mx^2}{r^3} + \frac{4x^2}{r^2}\left(\frac{m}{r}\right)^2\right\} \frac{dx^2}{c^2}$$

for $m \ll r$

$$\approx \left\{1 + \frac{2m}{r} + \frac{2mx^2}{r^3}\right\} \frac{dx^2}{c^2}$$

Thus, we obtain

$$dt = \left(1 + \frac{2m}{r} + \frac{2mx^2}{r^3}\right)^{\frac{1}{2}} \frac{dx}{c}$$

or

$$dt \cong \left(1 + \frac{m}{r} + \frac{mx^2}{r^3}\right)\frac{dx}{c} \qquad (25)$$

where $r^2 = x^2 + d^2$

where, in taking the last step we have again employed the fact that $m \ll r$.

Equation (25) can now be used to calculate the round trip transit time

$$\Delta t = 2\int_{-x_e}^{x_p} dt$$

$$= \frac{2}{c}\int_{-x_e}^{x_p} \left(1 + \frac{m}{r} + \frac{m\,x^2}{r^3}\right) dx$$

$$= \frac{2}{c}\left(x_p + x_e\right) + \frac{2m}{c}\int_{-x_e}^{x_p} \frac{dx}{r} + \frac{2m}{c}\int_{-x_e}^{x_p} \frac{x^2 dx}{r^3}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\frac{2m}{c}\ln\left(\frac{r}{d} + \frac{x}{d}\right)\Bigg|_{-x_e}^{x_p} \qquad \frac{2m}{c}\ln\left(\frac{r}{d} + \frac{x}{d}\right)\Bigg|$$

$$+\frac{2m}{c}\left(\frac{x}{r}\right)\Bigg|_{-x_e}^{x_p}$$

Thus, the round trip transit time is given by

$$\Delta t = \frac{2}{c}\left(x_p + x_e\right) + \frac{4m}{c} \ln\left(\frac{r_p + x_p}{r_e - x_e}\right) + \frac{2m}{c}\left(\frac{x_p}{r_p} + \frac{x_e}{r_e}\right) \tag{26}$$

The significant measureable quantity, which would provide a test of General Relativity would be the difference, if any, between round trip transit times measured when the earth and an inner planet are at orientation for which calculated from (26) is at a maximum and minimum.

The three extreme relative positions are illustrated in Figures (3 a,b,c)

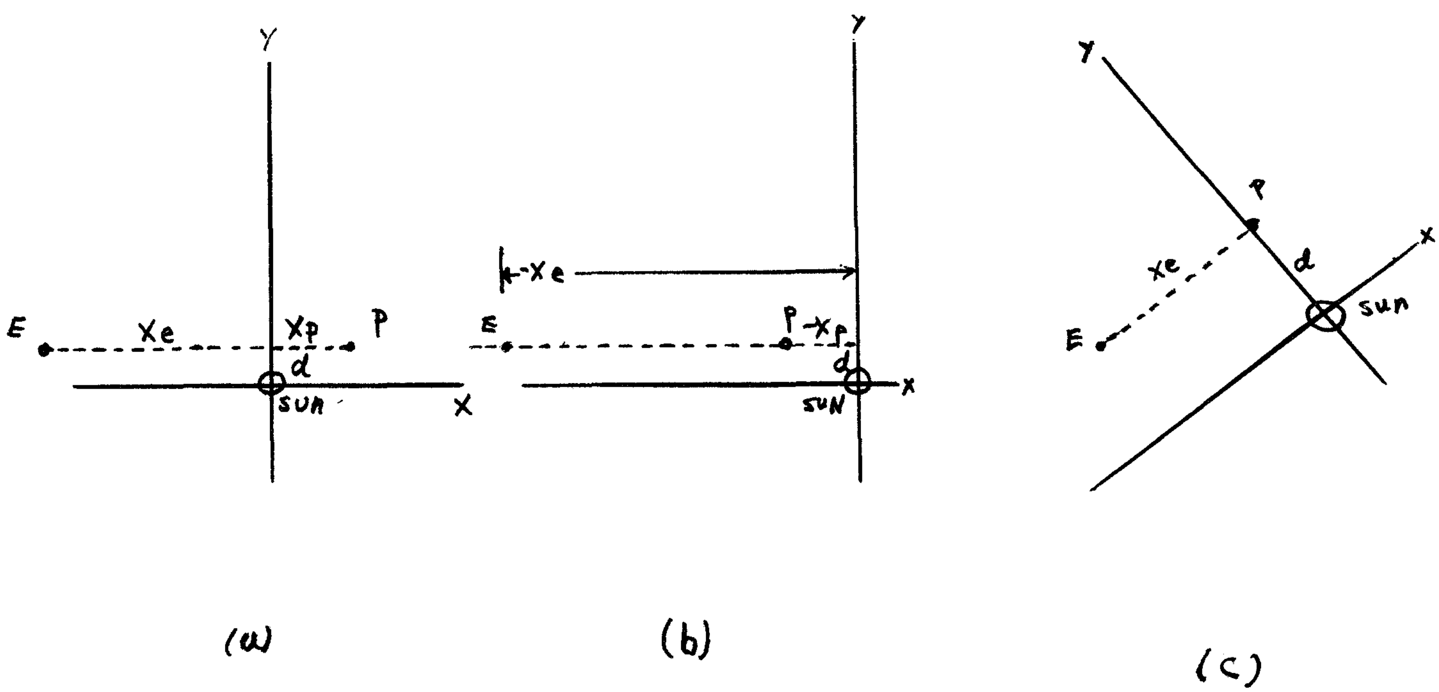


fig (3)

Figure (3a) is meant to represent the condition that the earth and planet are almost at superior conjunction, with the planet almost exactly on the opposite side of the sun as the earth. In which case:

$$d \ll x_e$$

$$d \ll x_p$$

and (26) simplifies to

$$\Delta t \approx \frac{2}{c}(x_p + x_e) + \frac{4m}{c} \ln\left(\frac{x_p + \frac{1}{2}\frac{d^2}{x_p} + x_p}{x_e + \frac{1}{2}\frac{d^2}{x_e} - x_e} \right)$$

which gives

$$\Delta t_a \approx \frac{2}{c}(x_p + x_e) + \frac{4m}{c} \ln\left(\frac{4x_e\, x_p}{d^2}\right) \qquad (28a)$$

Inferior conjunction, when the planet is almost exactly on a line between the sun and earth is pictured in Figure (3b). Here again

$$d \ll x_e, x_p \quad .$$

However, the sign of x_p is now reversed and we obtain for the transit time, from (27)

$$\Delta t_b \approx \frac{2}{c}(x_e - x_p) + \frac{4m}{c} \ln \frac{x_e}{x_p} \tag{28b}$$

Finally, in Figure (3c) the planet as viewed from the earth is furthest to the east (or west) of the sun, in this case:

$$x_p \approx 0$$
$$d \ll x_e$$

These conditions can only be realized for Mercury since $d \approx x_e$ for Venus at elongation.

For Mercury, we get

$$\Delta t_c = \frac{2}{c}(x_e) + \frac{4m}{c} \ln \frac{2x_e}{d} + \frac{2m}{c} \tag{28c}$$

The first terms in each of 28a, 28b, and 28c are recognized as the flat space transit times. When these are subtracted we obtain the variations from flat space values given by General Relativity. These are:

$$\begin{aligned} \delta t_a &= \frac{4m}{c} \ln \frac{4x_e x_p}{d^2} \\ \delta t_b &= \frac{4m}{c} \ln \frac{x_e}{x_r} \\ \delta t_c &= \frac{4m}{c} \ln \left(\frac{2x_e}{d} + \frac{1}{2} \right) \end{aligned} \tag{29}$$

Theoretical calculations of δt_a, δt_b and δt_c for Mercury yield the result that

$$\delta t_a \approx 1.6 \times 10^{-4} \text{ sec}$$

$$d \approx 2\ R_s$$

$$\delta t_b \approx \delta t_c \approx .1 \times 10^{-4} \text{ sec.}$$

Thus the close approach of the beam to the sun, in a, considerably slows down the transit time. This departure from flat space is outside experimental error of present equipment. Thus Shapiro suggests that a measurement of δt_a would provide a test of general relativity.

STATIC INTERIOR SOLUTION

If the matter supports no transverse stresses and has no mass motion, then its energy momentum tensor is given by

$$T_1^{\,1} = T_2^{\,2} = T_3^{\,3} = -P\,, \quad T_4^{\,4} = \rho$$

where P and ρ are respectively the pressure and energy density measured in proper coordinates. If we write the most general spherically symmetric static line element as

$$ds^2 = -e^{\lambda(r)}dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\;d\phi^2 + e^{\nu(r)}dt^2\,,$$

Einstein's field equations reduce to

$$8\pi P = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} \tag{1}$$

$$8\pi\rho = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} \tag{2}$$

$$\frac{dP}{dr} = -\frac{(P+\rho)}{2}\,\nu' \tag{3}$$

(primes denote differentiation with respect to r)

These three equations together with the equation of state of the material $\rho = \rho(P)$ determine the mechanical equilibrium of the mass distribution as well as the dependence of the $g_{\mu\nu}$'s on r.

The boundary of the mass distribution is determined by the conditions $r = r_b$ for $P = 0$ and $P > 0$ for $r < r_b$. For $r > r_b$ we have that $P = \rho = 0$ and Schwarzschild's exterior solution gives

$$e^{-\lambda(r)} = 1 - \frac{2m}{R} , \quad e^{\nu(r)} = 1 - \frac{2m}{R}$$

where m is the total Newtonian mass of the matter as calculated by a distant observer.

Integrating equation (3) we obtain

$$\nu(r) = \nu(r_b) - \int_0^{P(r)} \frac{2\,dP}{P + \rho(P)}$$

or

$$e^{\nu(r)} = e^{\nu(r_b)} \exp - \int_0^{P(r)} \frac{2\,dP}{P + \rho(P)} .$$

The constant $e^{\nu(r_b)}$ is determined by making e^{ν} continuous across the boundary. Then

$$e^{\nu(r)} = \left(1 - \frac{2m}{r_b}\right) \exp - \int_0^{P(r)} \frac{2\,dP}{P + \rho(P)} .$$

Thus $e^{\nu(r)}$ is known as a function of r if P is known as a function of r. In equation (2) introduce the new variable

$$U(r) = 1/2\ r(1 - e^{-\lambda}) \quad .$$

Then equation (2) becomes

$$\frac{dU}{dr} = 4\pi \rho(P)\ r^2 \quad . \tag{4}$$

Using this new variable and equation (3), equation (1) may be written as

$$\frac{dP}{dr} = -\frac{P + \rho\,(P)}{r(r-2U)}\ (4\pi \rho r^3 + U) \tag{5}$$

These two equations form a system of two first-order equations in U and P. Starting with some initial values $U = U_o$ and $P = P_o$ at $r = 0$, the two equations are integrated simultaneously to the value $r = rb$ where $P = 0$. The value of $U(r)$ at $r = rb$ is determined by joining $e^{\lambda(r)}$ continuously across the boundary. We find that

$$U(r_b) = \frac{r_b}{2}\left[1 - e^{\lambda(r_b)}\right]$$

$$= \frac{r_b}{2}\left[1 - \quad 1 - \frac{2m}{r_b}\right] = m \quad .$$

Now consider the special case that ρ = constant. Then the equation

$$\frac{dU}{dr} = 4\pi \rho\ r^2$$

is easily integrated and we have

$$U = \frac{4}{3} \pi \rho r^3 + c$$

$$= 1/2\ r\ (1 - e^{-\lambda})$$

or

$$e^{-\lambda(r)} = 1 - \frac{8\pi\rho}{3} r^2 + \frac{c}{r}$$

In order that this solution be continuous across the boundary we require that $c = 0$. Then $e^{-\lambda(r)} = 1 - (r/R)^2$ where

$$R = \sqrt{3} / \sqrt{8\pi\rho}$$

Equation (3) can be written as

$$\frac{dP}{P+\rho} = -\frac{1}{2}\, d\nu\,(r)$$

and its solution is

$$P + \rho = \text{constant} \times e^{-\frac{1}{2}\nu(r)} .$$

Substituting equations (1) and (2) into this expression we have

$$e^{\frac{1}{2}\nu(r)}\ e^{-\lambda(r)} \left(\frac{\lambda'}{r} + \frac{\nu'}{r} \right) = \text{constant} .$$

The solution of this equation can be written as

$$e^{\frac{1}{2}\nu(r)} = A - B\ (1 - r^2/R^2)^{\frac{1}{2}}$$

where A and B are constants.

Now we have that

$$ds^2 = -\frac{dr^2}{1-r^2/R^2} - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2$$

$$+ \left[A - B\,(1-r^2/R^2)^{\frac{1}{2}}\right]^2 dt^2$$

and

$$8\pi\rho = \frac{1}{R^2}\,\frac{3B(1-r^2/R^2)^{\frac{1}{2}} - A}{A - B\,(1-r^2/R^2)^{\frac{1}{2}}}$$

The constants A and B may be determined by joining these equations to the exterior solutions at $r = r_b$,

$$ds^2 = \frac{-dr^2}{1-\frac{2m}{r_b}} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + (1 - \frac{2m}{r_b})\, dt^2$$

and

$$P = 0 .$$

We find that

$$A = \frac{3}{2}\,(1 - r_b^2/R^2)^{\frac{1}{2}} ,$$

$$B = 1/2 , \quad \text{and} \quad m = \frac{4\pi}{3}\rho\, r_b^3$$

and for the interior

$$ds^2 = \frac{-\,dr^2}{1-r^2/R^2} - r^2\,(d\,\theta^2 + \sin^2\theta\, d\phi^2)$$

$$+ \left[3/2\ (1-r_b{}^2/R^2)^{\frac{1}{2}} - 1/2 \left(1-r^2/R^2\right)^{\frac{1}{2}} \right]^2 dt^2$$

In order that the pressure remains finite and positive we require that

$$A - B\ (1-r^2/R^2\ )^{1/2} > 0 \qquad .$$

Since $r \geq 0$ this reduces to

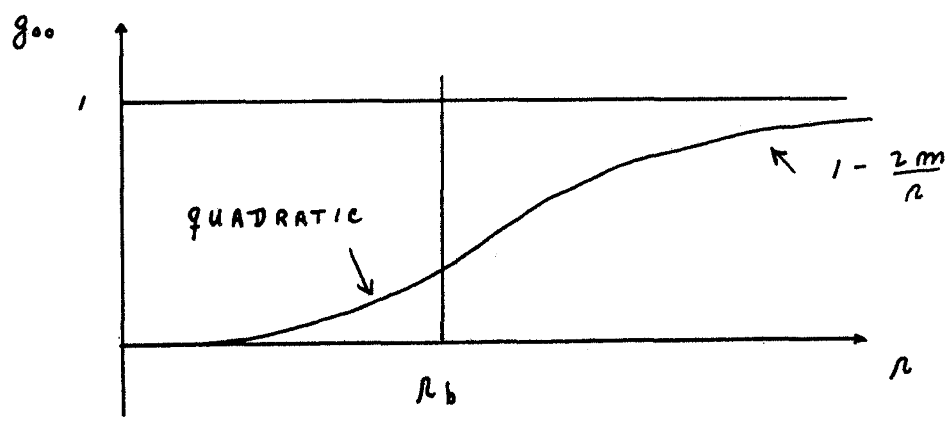
$$A - B \geq 0$$

$$\frac{3}{2}\ (1-r_b{}^2/R^2)^{\frac{1}{2}} - 1/2 > 0$$

or

$$\frac{r_b{}^2}{R^2} < \frac{8}{9}\ .$$

The dependence of g_{00} on r is illustrated in the following graph.



Neutron Stars

Since the matter density of a neutron star is very high ($\sim 10^{16}$ gm/cm^3), Oppenheimer and Volkoff (Phys. Rev. 55, 374 (1939)) have considered a model in which the equation of state is that of an ideal or degenerate Fermi gas at zero temperature.

In order to discuss this model we consider the equations discussed above for the case of an ideal Fermi gas at zero temperature. The relevant equations are

$$\frac{dP}{dr} = - \frac{(P + \rho)(U + 4\pi r^3 \rho)}{r^2 (1 - 2\, U/r)} \,, \tag{5}$$

$$\frac{dU}{dr} = 4\pi r^2 \rho \,. \tag{6}$$

For an ideal Fermi gas we have

$$\rho = \frac{2\,I + 1}{h^3} \int_0^{p_f} \epsilon \, d^3 p \,,$$

$$P = \frac{2\,I + 1}{h^3} \, \frac{1}{3} \int_0^{p_f} p \, \frac{d\epsilon}{dp} \, d^3 p \,,$$

$$\epsilon = (p^2 c^2 + m^2 c^4)^{\frac{1}{2}} \,.$$

If we introduce the variable

$$x = p_f / mc$$

these equations can be written as

$$\rho = \frac{m^4 c^5}{24 \pi^2 h^3} \Big\{ 8 x^3 (x^2 + 1)^{\frac{1}{2}}$$

$$- x(2 x^2 - 3)(x^2 + 1)^{\frac{1}{2}} + 3 \sin h^{-1} x \Big\}$$

$$P = \frac{m^4 c^5}{24 \; ^2 h^3} \Big[x(2x^2 - 3)(x^2 + 1)^{\frac{1}{2}} + 3 \sin h^{-1} x \Big]$$

If we now introduce another variable, t, such that

$$x = \sin h\,(t/2)$$

we have

$$\rho = k\,(\sin h\, t - t)\;,$$

$$P = \frac{1}{3} K\,(\sin h\, t - 8 \sin h\, t/2 + 3\, t)\;,$$

where

$$K = \frac{\pi m^4 c^5}{4 h^3} \; .$$

Thus we have

$$P \propto \rho^{5/3} \;, \quad t \rightarrow 0$$

$$P \propto \rho \;, \quad t \rightarrow \infty$$

If we substitute this equation of state into the expressions obtained from the field equations (Eqs. 4 and 5) we find that

$$\frac{dU}{dr} = 4\pi\ r^2\ K\ (\sin h\ t - t)$$

$$\frac{dt}{dr} = -\frac{4}{r(r-2U)}\ \frac{\sin h\ t - 2\ \sin h\ t/2}{\cos h\ t - 4\ \cos h\ t/2 + 3}$$

$$\left[\frac{4}{3}\pi\ K\ r^3\ (\sin h\ t - 8\ \sin h\ t/2 + 3t) + U\right]$$

These equations are to be integrated from the values $U = 0$, $t = t_o$ at $r = 0$ to $r = r_b$ where $t_b = 0$ (which makes $P = 0$) and $U = U_b$.

The units which we have been using are such that $c = 1$ and $G = 1$. This determines the unit of time and the unit of length. This unit of length may now be fixed by requiring that $K = 1/4\pi$.

The new unit of length is

$$a = \frac{1}{\pi}\left(\frac{h}{mc}\right)^{3/2}\ \frac{c}{(mG)^{\frac{1}{2}}}$$

while the unit of mass is

$$b = \frac{c^2}{G}\ a$$

For a neutron gas

$$a = 1.36 \times 10^{6} \text{ cm}$$

$$b = 1.83 \times 10^{34} \text{ gm} \quad .$$

It is only possible to find analytical solutions to these equations for $t_o \rightarrow 0$. Then the equations reduce to

$$\frac{dU}{dr} = \frac{1}{2} r^2 e^t$$

and

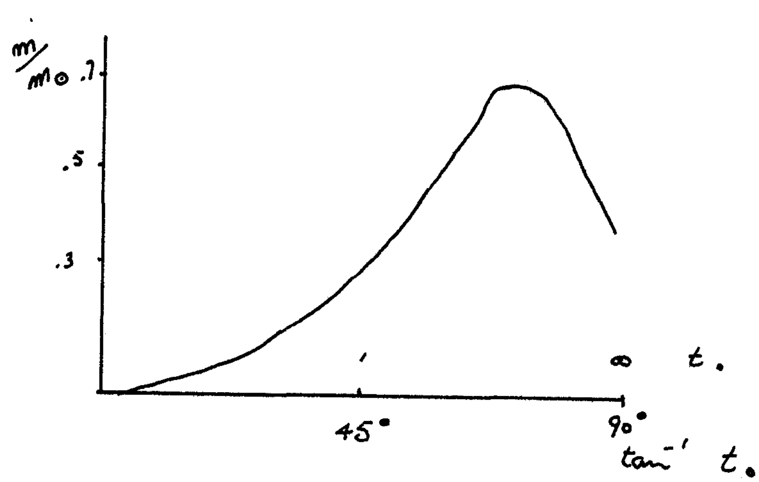
$$\frac{dt}{dr} = \frac{4}{r(r-2U)} \left(\frac{r^3}{6} e^t + U \right)$$

which have solutions of the form

$$e^t = 3/7 \; r^2 \quad \text{and} \quad U = 3 \; r/14$$

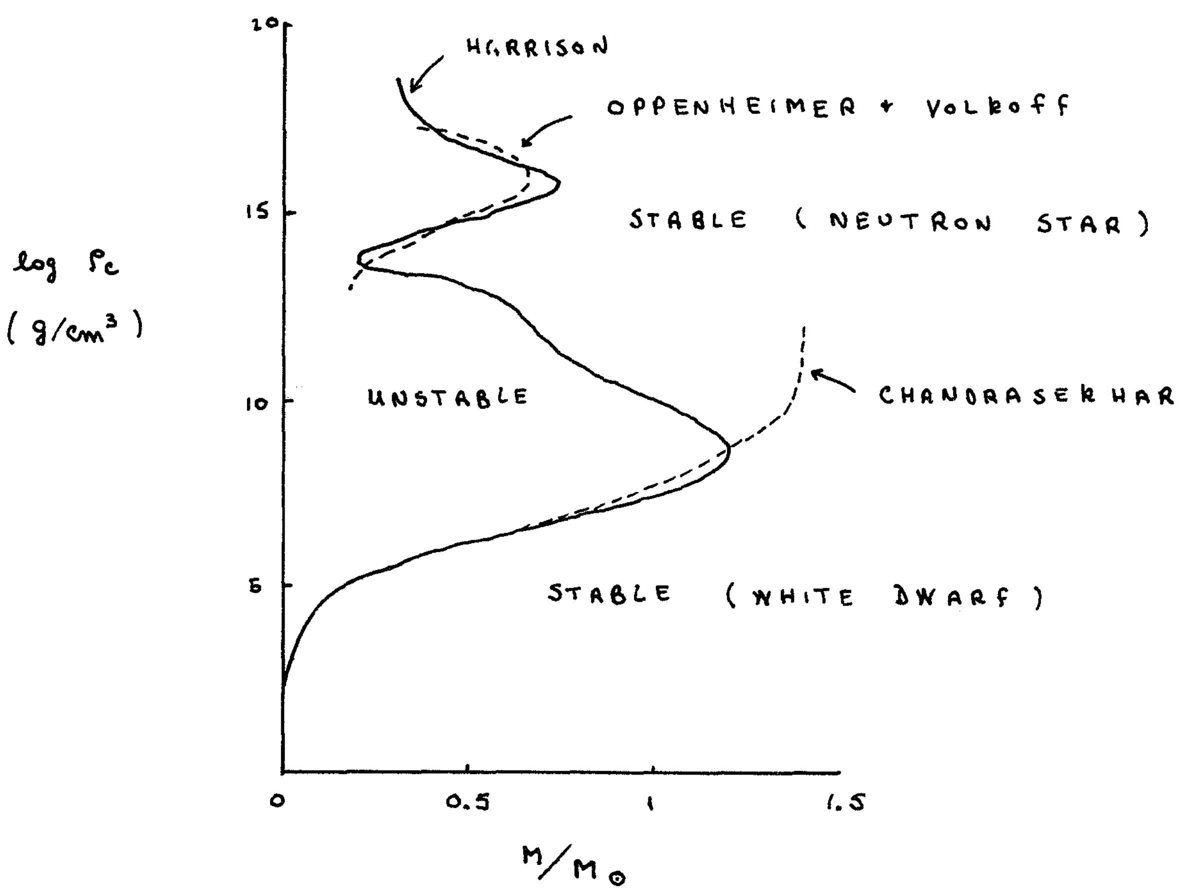
corresponding to the boundary conditions $t_o = \infty$ and $U_o = 0$.

Oppenheimer and Volkoff (Phys. Rev. 55, 374 (1939)) have integrated these equations numerically. Their results are summarized in the plot of m vs. $\tan^{-1} t_o$ which follows.

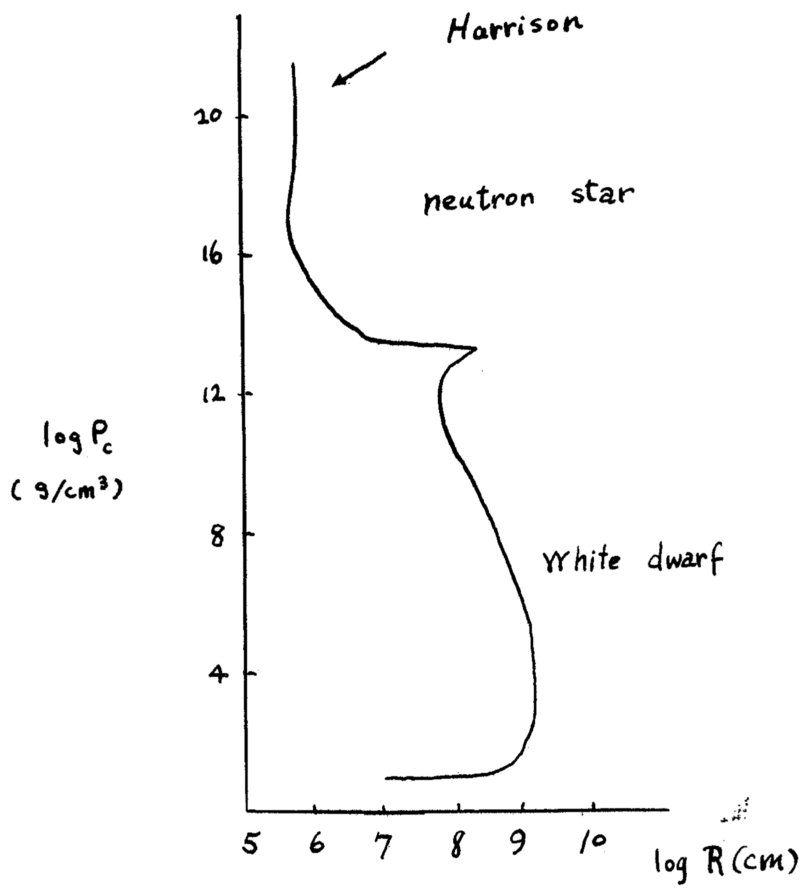


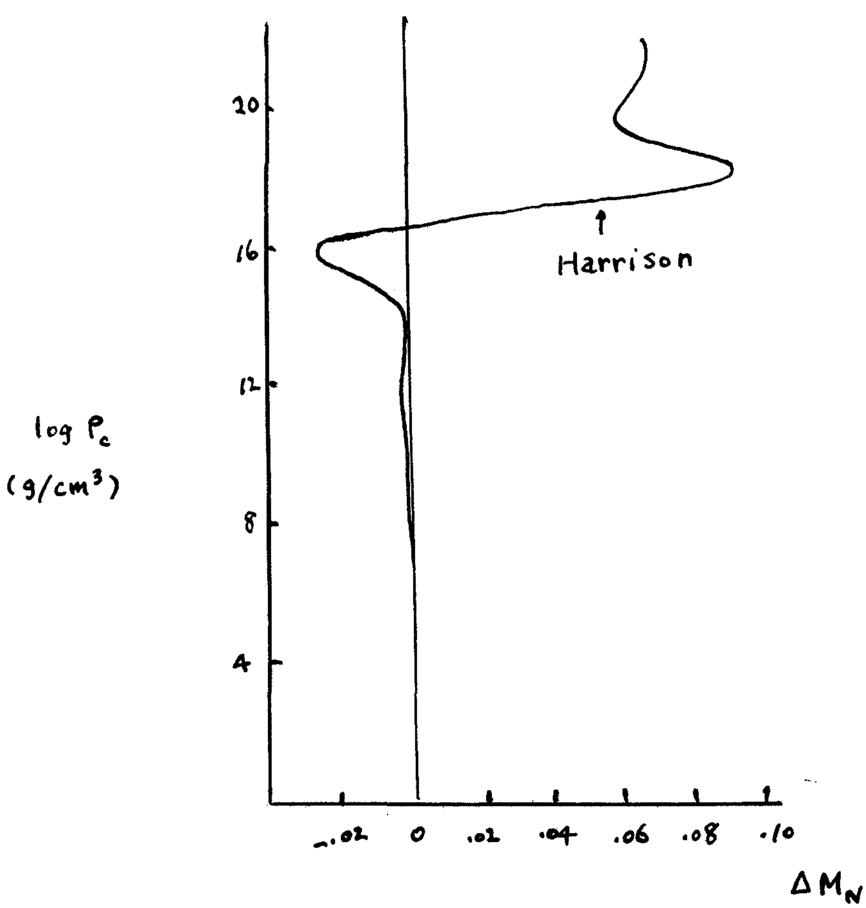
This shows that for this model no static solutions exist for $m > 3/4 \odot$.

Calculations using a more realistic equation of state have been performed by B. K. Harrison (Phys. Rev. 137, B1644 (1965)). The results he obtained are given in the following figures. (The results of some other calculations are also given for comparison.)



THE MASS - CENTRAL DENSITY RELATION FOR ZERO TEMPERATURE STARS





This is a plot of the average mass per nucleon, $M(R)/N(R)$, on a scale which shows the deviation from the low-density value; the variable

$$\Delta M_N = \frac{M(R)}{N(R)}\left[\frac{N(R)}{M(R)}\right]_{\substack{\text{low}\\\text{density}}} - 1$$

is the binding energy per nucleon in units of nucleon rest energy.

Supernova and Neutron Stars

For a star of central temperature about 10^9 °K, the core is composed almost entirely of Si^{28}. At this temperature the neutrino processes become important and it is possible for the star to dissipate energy at the rate of 10^{15} erg/cm^3-sec . Since very little energy is generated in the transformation $Si^{28} \rightarrow Fe^{56}$ at about $1\text{-}3 \times 10^9$ °K , energy can be supplied only by the gravitational contraction of the core. The star can now either cool down to form a white dwarf or undergo gravitational contraction. Contraction of the core will increase the central density and temperature and at about $T = 6 \times 10^9$ °K either the neutrino processes or this transformation of Fe^{56} to He^4 will cause instabilities and collapse.

For a massive star (say $M = 20\ M_\odot$) to form a cold star it is necessary that most of its matter is ejected during the collapse of the core. This must occur because no cold star can exist with a mass greater than about $1.4\ M_\odot$ (Chandrasekhar mass limit). For a star of mass M and radius R contracting to a mass $M_\odot$ and radius r , and ejecting a mass of $M - M_\odot$, the gravitational energy release must be about

$$\frac{G\,(M^2 - M_\odot^2)}{R^2}$$

This energy must be supplied by the contracting core, therefore

$$\frac{G\ (M^2 - M_\odot^2)}{R} \qquad \frac{GM_\odot^2}{r^2}$$

If $M = 20\ M_\odot$ and $R = 10^9$ cm we find that $r \sim \frac{R}{400}$, thus the density of the core will be 10^6 higher than that of the star before collapse or about $10^{13}\ \mathrm{gm/cm^3}$ which is the density of a neutron star.

We can now apply the theory of an ideal Fermi gas to this neutron star to find its thermal energy. The specific heat per unit mass is

$$C_v = \frac{\pi^2 k}{m_n} \left(\frac{kT}{m_n C^2} \right) \frac{x(x^2 + 1)^{\frac{1}{2}}}{x^3}$$

$$= .75 \times 10^{-4}\ T\ \frac{x(x^2 + 1)^{\frac{1}{2}}}{x^3}$$

where $x = \frac{P_F}{m_n C^2}$ and P_F is the Fermi momentum for the neutron gas. The thermal energy per unit mass is

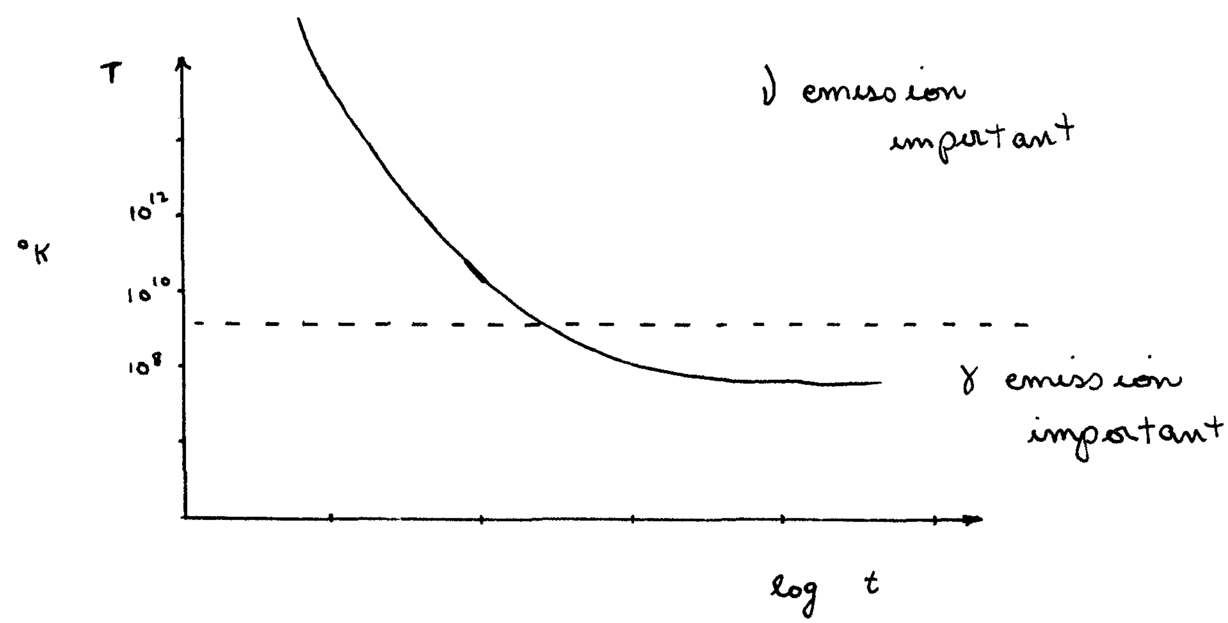
$$= .38 \times 10^{-4}\ T^2\ \frac{x(x^2 + 1)^{\frac{1}{2}}}{x^3}$$

For $T = 10^9\ {}^oK$ and $x \sim 1$ we have

$$\epsilon \sim 4 \times 10^{13} \text{ ergs/gm}$$

Therefore the total thermal energy of a neutron star will be about $M \times 10^{14} \sim 10^{47}$ ergs.

Neutron stars could be formed from the collapse of massive stars with the release of a large amount of energy. The only observed objects which emit this kind of energy are supernova, and therefore, it is thought that neutron stars are the results of supernova explosions. The temperature of a neutron star as a function of the time after the supernova is:



t = days after supernova -

Within a short time after the supernova the main source of energy release for the neutron star will be γ emission. The total energy radiated per second will be

$$L = 4\pi R \frac{ac}{4} T_s^4$$

where T_s is the surface temperature, R is the radius of the star, and $\frac{ac}{4}$ is the Stefan-Boltzmann constant. For $R \sim 10$ km and $T_s \sim 10^7$ °K we find that $L \sim 10^{38}$ erg/sec. The lifetime will approximately be given by

$$\frac{\text{Total energy}}{L} = \frac{10^{47}\text{ergs}}{10^{38}\text{ergs/sec}}$$

$$= 10^9 \text{ sec}$$

or about 100 years. But as T decreases, L also decreases, so the lifetime can be expected to be much greater than 100 years.

Although the luminosity of neutron stars may be 10^{38} erg/sec, they would not be detectable from the earth due to absorption in the atmosphere. Chiu (Annals of Physics, 26, 364 (1964)) estimates that the luminosity of an average neutron star observed on earth would be about 10^{-6} $L_\odot$ ($L_\odot \sim 10^{33}$ erg/sec). To be detectable, such an object would have to be almost inside the solar system. It should be possible, however, to detect neutron stars from earth satellites.

NON-STATIC SOLUTION OF THE FIELD EQUATIONS

We now consider the field equations for a distribution of matter which is in motion. For the case of a perfect fluid with no radiation present the energy-momentum tensor is

$$T_\alpha{}^\beta = - g_\alpha{}^\beta \; P + (P + \rho) \frac{dx_\alpha}{ds} \frac{dx^\beta}{ds} \quad .$$

$$\epsilon = \rho c^2 = \rho \quad (c = 1)$$

(See the section entitled Classical Field Theory, page 13.)
If we restrict ourselves to spherical symmetry, i.e., the velocity at each point must be directed along the radius, we have that

$$\frac{dx_\alpha}{ds} = \frac{dx^\alpha}{ds} = 0 \qquad \text{for } \alpha = 2, 3$$

where $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, and $x^4 = t$.

Using the spherically symmetric line element we had above,

$$ds^2 = -e^{+\lambda} \; dr^2 - r^2 \, d\theta^2 - r^2 \sin^2\theta \; d\phi^2 + e^\nu \; dt^2$$

with

$$\lambda = \lambda \, (r, t)$$

and

$$\nu = \nu (r, t)$$

we have

$$g_{\alpha\beta} = \begin{matrix} -e^{\lambda} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin^2\theta & 0 \\ 0 & 0 & 0 & e^{\nu} \end{matrix}$$

and

$$g_{\alpha}{}^{\beta} = \delta_{\alpha}{}^{\beta} \quad .$$

Then the field equations can be written as follows:

$$\frac{8\pi G}{c^4} T_1{}^1 = -e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2}$$

$$\frac{8\pi G}{c^4} T_2{}^2 = \frac{8\pi G}{c^4} T_3{}^3 =$$

$$- e^{-\lambda}\left(\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r}\right)$$

$$+ \frac{e^{-\nu}}{2}\left(\ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda}\dot{\nu}}{2}\right)$$

$$\frac{8\pi G}{c^4} T_4{}^4 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}$$

$$\frac{8\pi G}{c^4} T_1{}^4 = e^{-\nu} \frac{\dot{\lambda}}{r} \quad ,$$

$$\frac{8\pi G}{c^4} T_4{}^1 = - e^{-\lambda} \frac{\dot{\lambda}}{r} \quad ;$$

where the primes represent differentiation with respect to r and the dots differentiate with respect to t. We also have

$$T_1{}^1 = (P + \rho) \frac{dx_1}{ds} \frac{dx^1}{ds} - P \quad ,$$

$$T_2{}^2 = T_3{}^3 = - P \quad ,$$

$$T_4{}^4 = (P + \rho) \frac{dx_4}{ds} \frac{dx^4}{ds} - P \quad ,$$

$$T_1{}^4 = (P + \rho) \frac{dx_1}{ds} \frac{dx^4}{ds}$$

$$T_4{}^1 = (P + \rho) \frac{dx_4}{ds} \frac{dx^1}{ds}$$

$$= - e^{(\nu-\lambda)/2} T_1{}^4 \quad .$$

With $P = 0$ we have the free gravitational collapse of matter. The general features of the solution obtained with $P = 0$ will apply even to the case that $P \neq 0$, provided the mass is great enough to cause collapse. In order to solve the problems we adopt a coordinate system which is comoving with the matter and take a line element of the form,

$$ds^2 = d\tau^2 - e^{\bar{\omega}(R,\tau)} dR^2 - e^{\omega(R,\tau)} (d\theta^2 + \sin^2\theta \; d\phi^2) \; .$$

Because the coordinates are comoving with the matter and the pressure is zero,

$$T_4^{\;4} = \rho$$

and all other components of the energy momentum tensor vanish.

In the comoving coordinates the field equations are:

$$8\pi \; T_1^{\;1} = 0 = e^{-\omega} - e^{-\bar{\omega}} \; \frac{\omega'^2}{4} + \ddot{\omega} + \frac{3}{4} \dot{\omega}^2 \; , \tag{1}$$

$$\begin{aligned} 8\pi \; T_2^{\;2} = 8\pi \; T_3^{\;3} = 0 = &-e^{-\bar{\omega}} \left(\frac{\omega''}{2} + \frac{\omega'^2}{4} + \frac{\bar{\omega}' - \omega'}{4} \right) \\ &+ \frac{\ddot{\bar{\omega}}}{2} + \frac{\dot{\bar{\omega}}^2}{4} + \frac{\ddot{\omega}}{2} + \frac{\dot{\omega}^2}{4} + \frac{\dot{\omega}\dot{\bar{\omega}}}{4} \; , \end{aligned} \tag{2}$$

$$\begin{aligned} 8\pi T_4^{\;4} = 8\pi\rho = e^{-\omega} - e^{-\bar{\omega}} &\left(\omega'' + \frac{3}{4} \omega'^2 - \frac{\bar{\omega}'\omega'}{2} \right) \\ &+ \frac{\dot{\omega}^2}{4} + \frac{\dot{\bar{\omega}}\dot{\omega}}{2} \; , \end{aligned} \tag{3}$$

$$8\pi \; e^{\bar{\omega}} \; T_4^{\;1} = -8\pi T_1^{\;4} = 0 = \frac{\omega'\dot{\omega}}{2} - \frac{\dot{\bar{\omega}}\omega'}{2} + \dot{\omega}' \tag{4}$$

with the primes and dots here representing differentiation with respect to R and τ respectively. Equation (4) has been integrated by Tolman and he finds that the solution is

$$e^{\bar{\omega}} = e^{\omega} \omega'^2 / 4 f^2(R) \tag{5}$$

where $f^2(R)$ is a positive but otherwise arbitrary function of R. We can find a sufficiently wide class of solutions if we put $f^2(R) = 1$.

Substituting equation (5) into equation (1) we have

$$\ddot{\omega} + \frac{3}{4} \dot{\omega}^2 = 0 . \tag{6}$$

The solution of this equation is

$$e^{\omega} = \left[F(R) \tau + G(R)\right]^{4/3} \tag{7}$$

where F and G are arbitrary functions of R. From equations (3), (5) and (7) we find that

$$8\pi\rho = \frac{4}{3} (\tau + G/F)^{-1} (\tau + G'/F')^{-1} \tag{8}$$

Substituting equation (5) into (2) also leads to equation (6), therefore equations (5), (7) and (8) are the solutions of the field equations.

If we now choose $G = R^{3/2}$, then at $\tau = 0$ we have

$$F F' = 9\pi R^2 \rho (R, \tau = 0)$$

We now consider the particular case that $\rho(R, \tau = 0)$ is constant and we have

$$\begin{aligned} F F' &= 9\pi \rho_0 R^2 , & R &< R_b \\ &= 0 , & R &> R_b \end{aligned}$$

where R_b is the radius of the boundary of the mass distribution. A particular solution of this equation is:

$$\begin{aligned} F &= - \frac{3}{2} r_o^{\frac{1}{2}} (R/R_b)^{3/2} , & R &< R_b \\ &= - \frac{3}{2} r_o^{\frac{1}{2}} , & R &> R_b \end{aligned}$$

where

$$r_o = \frac{8\pi}{3} \rho_0 R_b = 2 M .$$

We now transform this solution into the stationary coordinates which have a line element of the form

$$ds^2 = - e^{\lambda} \; d\, r^2 - r^2 \; (d\theta^2 + \sin^2\theta \;\; d\phi^2) + e^{\nu} \; dt^2 \quad .$$

Therefore

$$e^{\omega/2} = (F\tau + G)^{2/3} = r$$

and

$$\tilde{g}^{\mu\nu} = \sum_{\alpha\beta} \frac{\partial \tilde{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \tilde{x}^{\nu}}{\partial x^{\beta}} g^{\alpha\beta}$$

where $\tilde{x}' \to (r, \theta , \phi , t)$ and $x \to (R, \theta , \phi , \tau)$. We find that

$$\tilde{g}^{44} = e^{-\nu} = \dot{t}^2 - t'^2 / r'^2$$

$$= \dot{t}^2 \; (1 - \dot{r}^2) \; ,$$

$$\tilde{g}^{11} = - e^{-\lambda} = - (1 - \dot{r}^2) \quad ,$$

$$\tilde{g}^{14} = 0 = \dot{t}\, \dot{r} - t' / r' \quad ,$$

where the primes and dots still refer to differentiation with respect to R and τ respectively. Using the values of r, F, and G given above we find:

$$t'/t = \dot{r}\, r' = -(r_o R)^{\frac{1}{2}} \left[R^{3/2} - \frac{3}{2} r_o^{\frac{1}{2}} \tau \right]^{-2/3}, \quad R > R_b$$

$$= - r_o^{\frac{1}{2}} R R_b^{-3/2} \left[1 - 3/2\, r_o^{\frac{1}{2}} \tau R_b^{-3/2} \right]^{1/3}, \quad R < R_b \ .$$

The general solution of this equation is

$t = L\,(x)$ for $R > R_b$

$$\text{with} \quad x = \frac{2}{3 r_o^{\frac{1}{2}}} (R^{3/2} - r^{3/2}) - 2(r\, r_o)^{\frac{1}{2}} + r_o \ln \frac{r^{\frac{1}{2}} + r_o^{\frac{1}{2}}}{r^{\frac{1}{2}} - r_o^{\frac{1}{2}}}$$

$t = M\,(y)$ for $R < R_b$

$$\text{with} \quad y = \frac{1}{2} \left[(R/R_b)^2 - 1 \right] + \frac{R_b\, r}{r_o R},$$

where L and M are arbitrary functions of x and y respectively.

For $R > R_b$ we have that

$$e^{\lambda} = (1 - r_o/r)^{-1}$$

$$e^{\nu} = (1 - r_o/r) \ .$$

(These equations were derived above for the case of a static mass distribution, but they also hold outside a non-static mass.

See Landau and Lifshitz, "The Classical Theory of Fields", p. 326.) With these equations and the equation for $\tilde{g}^{44}$ given above it can be shown that

$$L(x) = t = x .$$

The requirement that $L = M$ for all τ, when $R = R_b$, gives

$$t = M(y) = \frac{2}{3} r_o^{-\frac{1}{2}} \left(R_b^{3/2} - r_o^{3/2} y^{3/2} \right) - 2 r_o y^{\frac{1}{2}} + r_o \ln \frac{y^{\frac{1}{2}} + 1}{y^{\frac{1}{2}} - 1} .$$

We now consider the asymptotic behavior of t. As $y \to 1$, $t \to \infty$ and we can write for $R < R_b$,

$$t \sim r_o \ln \frac{y^{\frac{1}{2}} + 1}{y^{\frac{1}{2}} - 1} ,$$

$$t \sim - r_o \ln (y - 1) ,$$

$$t \sim - r_o \ln \left\{ \tfrac{1}{2} \left[(R/R_b)^2 - 3 \right] + \frac{R_b}{r_o} \left(1 - \frac{3 r_o^{\frac{1}{2}} \tau}{2 R_b^{3/2}} \right)^{2/3} \right\} .$$

Thus t will be infinite for a finite value of τ, τ_*, such that

$$\tfrac{1}{2}\left(\frac{R}{R_b}\right)^2 - \frac{3}{2} + \frac{R_b}{r_o}\left(1 - \frac{3\, r_o^{\frac{1}{2}}\, \tau_o}{2\, R_b^{3/2}}\right)^{2/3} = 0$$

Therefore, after a time τ AN observeR comoving with the matter will be unable to send a light signal from the star; the cone within which a signal can escape has closed entirely. For the sun $r_o = 2.9$ km, $R_b = 10^{11}$ cm and $\tau_o \sim 10^5$ sec.

It can also be shown that for large values of t,

$$e^{-\lambda} \sim 1 - (R/R_b)^2 \left\{ e^{-t/r_o} + \tfrac{1}{2}\left[3 - (R/R_b)^2\right]\right\}^{-1}$$

$$e^{\nu} \sim e^{\lambda - 2t/r_o} \left\{ e^{-t/r_o} + \tfrac{1}{2}\left[3 - (R/R_b)^2\right]\right\}$$

(See Oppenheimer and Snyder Phys. Rev. 56, 455 (1939) for more details of the above discussion.)

It is also interesting to note that when

$$\tau = \tau_o = \frac{2}{3}\,\frac{R_b^{3/2}}{r_o^{\frac{1}{2}}}\left[1 - \left\{\frac{3}{2}\,\frac{r_o}{R_b} - \tfrac{1}{2}\left(\frac{R}{R_b}\right)^2 \frac{r_o}{R_b}\right\}^{3/2}\right],$$

$$r = \left[F\, \tau_o + G\right]^{2/3}$$

$$\Omega = \left\{ -\frac{3}{2} r_o^{\frac{1}{2}} \left(\frac{R}{R_b}\right)^{3/2} \left[\frac{3}{2} \frac{R_b^{3/2}}{r_o^{\frac{1}{2}}} \left\{ 1 - \left(\frac{3}{2} \frac{r_o}{R_b} - 1/2 \left(\frac{R}{R_b}\right)^2 \frac{r_o}{R_b} \right)^{3/2} \right\} \right] + R^{3/2} \right\}^{2/3},$$

$$= \frac{3}{2} \frac{R\, r_o}{R_b} - \frac{1}{2} \frac{R\, r_o}{R_b^{\,3}} \quad .$$

Thus we have that when $R = R_b$, $r = r_o$ when $\tau = \tau_0$. The point at which an observer comoving with the matter cannot send a light signal from the star, corresponds to the radius of the star reaching the Schwarzschild singularity, r_o , as measured by a distant observer.

In this discussion we have neglected radiation in assuming that

$$a T^4 \ll \rho c^2$$

or

$$T \ll T_r$$

where

$$T_r = \frac{\rho c^2}{a} \quad .$$

Some typical values of T_r are:

$M/$	$r_o = \frac{2GM}{c^2}$ (cm)	(r_o) (g/cm^3)	T_r (^oK)
1	2.96×10^5	1.84×10^{16}	$.7 \times 10^{10}$
10^2	2.96×10^7	1.84×10^{12}	$.7 \times 10^{12}$
10^4	2.96×10^9	1.84×10^8	$.7 \times 10^{11}$
10^6	2.96×10^{11}	1.84×10^4	$.7 \times 10^{10}$
10^8	2.96×10^{13}	$1.84 \times$	$.7 \times 10^9$
10^{10}	2.96×10^{15}	1.84×10^{-4}	$.7 \times 10^8$
10^{22}	2.96×10^{27}	1.84×10^{-28}	$.7 \times 10^2$

The Energy-Momentum Pseudo-Tensor

For flat space we have seen that the conservation laws of mechanics are given by the equations

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0$$

where $T^{\mu\nu}$ is the energy momentum tensor. The natural generalization of this result to a curved space time is

$$T^{\mu\nu}{}_{;\nu} = 0,$$

which has been used in constructing the field equations. We shall now investigate the conservation laws which are prescribed by this generalization.

The contracted covariant derivative can be written out to obtain

$$T^{\mu\nu}{}_{;\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^{\mu}_{\alpha\nu} T^{\alpha\nu} + \Gamma^{\nu}_{\alpha\nu} T^{\mu\alpha} = 0$$

or

$$T_\mu{}^\nu{}_{;\nu} = \frac{\partial T_\mu{}^\nu}{\partial x^\nu} - \Gamma^{\alpha}_{\mu\nu} T^{\nu}_{\alpha} + \Gamma^{\nu}_{\alpha\nu} T^{\alpha}_{\mu} = 0$$

Introducting the tensor density

$$\mathfrak{T}_\mu{}^\nu = T_\mu{}^\nu \sqrt{-g}$$

where g is the determinant of the metric tensor, and using the identities

$$\Gamma^{\sigma}_{\mu\sigma} = \frac{\partial}{\partial x^{\mu}} \log \sqrt{-g}$$

$$= \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} \sqrt{-g} ,$$

we can write

$$\frac{\partial T_{\mu}^{\nu}}{\partial x^{\nu}} = \frac{\partial}{\partial x^{\nu}} \frac{\mathfrak{T}_{\mu}^{\nu}}{\sqrt{-g}}$$

$$= \frac{1}{\sqrt{-g}} \frac{\partial \mathfrak{T}_{\mu}^{\nu}}{\partial x^{\nu}} + \mathfrak{T}_{\mu}^{\nu} \frac{\partial}{\partial x^{\nu}} \frac{1}{\sqrt{-g}}$$

$$= \frac{1}{\sqrt{-g}} \frac{\partial \mathfrak{T}_{\mu}^{\nu}}{\partial x^{\nu}} + \mathfrak{T}_{\mu}^{\nu} \frac{(-1/2)}{(-g)^{3/2}} \frac{\partial(-g)}{\partial x^{\nu}} .$$

Now we can express the equation

$$\frac{\partial T_{\mu}^{\nu}}{\partial x^{\nu}} - \Gamma^{\alpha}_{\mu\nu} T_{\alpha}^{\nu} + \Gamma^{\nu}_{\alpha\nu} T_{\mu}^{\alpha} = 0$$

as

$$\frac{1}{\sqrt{-g}} \frac{\partial \mathfrak{T}_{\mu}^{\nu}}{\partial x^{\nu}} + \mathfrak{T}_{\mu}^{\nu} \frac{(-1/2)}{(-g)^{3/2}} \frac{\partial(-g)}{\partial x^{\nu}}$$

$$- \Gamma^{\alpha}_{\mu\nu} \frac{\mathfrak{T}_{\alpha}^{\nu}}{\sqrt{-g}} + \frac{\mathfrak{T}_{\mu}^{\alpha}}{\sqrt{-g}} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\alpha}} \sqrt{-g} = 0 .$$

The second and fourth terms cancel in the above expression and

$$\frac{\partial \mathfrak{T}_{\mu}^{\nu}}{\partial x^{\nu}} - \Gamma^{\alpha}_{\mu\nu} \mathfrak{T}_{\alpha}^{\nu} = 0 .$$

Since

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left[\frac{\partial g_{\mu\beta}}{\partial x^{\nu}} + \frac{\partial g_{\nu\beta}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}} \right],$$

we obtain

$$\Gamma^{\alpha}_{\mu\nu} \mathfrak{T}^{\nu}_{\alpha} = \frac{\mathfrak{T}^{\nu\beta}}{2} \left[\frac{\partial g_{\mu\beta}}{\partial x^{\nu}} + \frac{\partial g_{\nu\beta}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}} \right]$$

AND

$$\frac{\partial \mathfrak{T}_{\mu}^{\nu}}{\partial x^{\nu}} = \frac{1}{2} \mathfrak{T}^{\nu\beta} \frac{\partial g_{\mu\beta}}{\partial x^{\nu}} + \frac{1}{2} \mathfrak{T}^{\nu\beta} \frac{\partial g_{\nu\beta}}{\partial x^{\mu}} - \frac{1}{2} \mathfrak{T}^{\nu\beta} \frac{\partial g_{\mu\nu}}{\partial x^{\beta}}.$$

This reduces to

$$\frac{\partial \mathfrak{T}_{\mu}^{\nu}}{\partial x^{\nu}} - \frac{1}{2} \mathfrak{T}^{\nu\beta} \frac{\partial g_{\nu\beta}}{\partial x^{\mu}} = 0$$

because

$$\mathfrak{T}^{\mu\nu} = \mathfrak{T}^{\nu\mu}.$$

As we are interested in obtaining conservation laws, we would like to have an equation of the form

divergence of something = 0.

This can be accomplished by writing

$$\frac{\partial}{\partial x^{\nu}} \left(\mathfrak{T}_{\mu}^{\nu} + t_{\mu}^{\nu} \right) = 0$$

where
$$\frac{\partial t_\mu^\nu}{\partial x^\nu} = -\frac{1}{2} \mathcal{T}^{\sigma\beta} \frac{\partial g_{\mu\sigma}}{\partial x^\beta} .$$

Thus we have reduced our original equation

$$T^{\mu\nu}{}_{;\nu} = 0$$

to the diveregence of $(\mathcal{T}_\mu^\nu + t_\mu^\nu)$ equals zero. We are still faced with the problem, however, of finding an expression for t_μ^ν . It can be shown (See Tolman, Relativity, Thermodynamics, and Cosmology, p. 222) that

$$t_\beta{}^\alpha = \frac{1}{16\pi} \left\{ \frac{\partial}{\partial x^\beta} (g^{\mu\nu} \sqrt{-g}) \left[- \Gamma^\alpha_{\mu\nu} + \frac{1}{2} g^\alpha_\mu \Gamma^\delta_{\nu\delta} + \frac{1}{2} g^\alpha_\nu \Gamma^\sigma_{\mu\sigma} \right] + g^\alpha_\beta \left[\sqrt{-g}\, g^{\mu\nu} (\Gamma^\epsilon_{\mu\gamma} \Gamma^\gamma_{\nu\epsilon} - \Gamma^\lambda_{\mu\nu} \Gamma^\omega_{\lambda\omega}) \right] + 2 g^\alpha_\beta \Lambda \sqrt{-g} \right\} .$$

Thus $t_\beta{}^\alpha$ is not a tensor but it is defined in any coordinate system by the above equation and may be called the energy-momentum pseudo-tensor. Therefore the equation

$$\frac{\partial}{\partial x^\nu} (\mathcal{T}_\mu^\nu + t_\mu^\nu) = 0 ,$$

although not a tensor equation, is still covariant because it will have the same form in all coordinate systems.

We now integrate this equation over the spatial volume, $a \leq x^1 \leq b$, $c \leq x^2 \leq d$, and $e \leq x^3 \leq f$, for a fixed value of the time x^4. Then we have

$$\int_e^f \int_c^d \int_a^b \frac{\partial}{\partial x^4} \left(\mathcal{T}_\mu^{\,4} + t_\mu^{\,4} \right) dx^1 dx^2 dx^3$$

$$= - \int_e^f \int_c^d \int_a^b \left\{ \frac{\partial}{\partial x^1} \left(\mathcal{T}_\mu^{\,1} + t_\mu^{\,1} \right) + \frac{\partial}{\partial x^2} \left(\mathcal{T}_\mu^{\,2} + t_\mu^{\,2} \right) \right.$$

$$\left. + \frac{\partial}{\partial x^3} \left(\mathcal{T}_\mu^{\,3} + t_\mu^{\,3} \right) \right\} dx^1 dx^2 dx^3 .$$

Carrying out some of the integrations on the right hand side we have

$$\frac{d}{dx^4} \int_e^f \int_c^d \int_a^b \left(\mathcal{T}_\mu^{\,4} + t_\mu^{\,4} \right) dx^1 dx^2 dx^3$$

$$= - \int_e^f \int_c^d \left[\mathcal{T}_\mu^{\,1} + t_\mu^{\,1} \right]_a^b dx^2 \, dx^3$$

$$- \int_e^f \int_a^b \left[\mathcal{T}_\mu^{\,2} + t_\mu^{\,2} \right]_c^d dx^1 \, dx^3 - \int_c^d \int_a^b \left[\mathcal{T}_\mu^{\,3} + t_\mu^{\,3} \right]_e^f dx^1 \, dx^2 .$$

For a flat space time $g \to -1$, $t_\mu^\nu \to 0$, and $\mathfrak{T}_\mu^\nu \longrightarrow T_\mu^\nu$

and the above equations become

$$\frac{d}{dx^4} \int_e^f \int_c^d \int_a^b T_\mu^4 \, dx^1 dx^2 dx^3$$

$$= - \int_e^f \int_c^d \left[T_\mu^1 \right]_a^b dx^2 dx^3 - \int_e^f \int_a^b \left[T_\mu^2 \right]_c^d dx^1 dx^3$$

$$- \int_c^d \int_a^b \left[T_\mu^3 \right]_e^f dx^1 dx^2$$

These equations represent the conservation of energy and momentum with

$$P_\mu = \int_e^f \int_c^d \int_a^b T_\mu^4 \, dx^1 dx^2 dx^3 ,$$

P_i = momentum, and P_4=energy

Therefore for non-flat space time we write the energy and momentum as

$$P_\mu = \int_e^f \int_c^d \int_a^b \left(\mathfrak{T}_\mu^4 + t_\mu^4 \right) dx^1 dx^2 dx^3$$

where $\mathfrak{T}_\mu^4$ represents the energy and momentum density of the matter and t_μ^4 that of the gravitational field.

WEAK FIELD APPROXIMATION - GRAVITATIONAL WAVES

1. The Wave Equation

In this section we shall study the Einstein Field Equations for the case in which the metric $g_{\mu\nu}$ differs from its Galilean value $\eta_{\mu\nu}$ by quantities of order ϵ ($\epsilon \ll 1$); that is, we expand the metric $g_{\mu\nu}$ about its Galilean values as a series in ϵ, viz.,

(1) $$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu} + O(\epsilon^2)$$

where $$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

In raising (or lowering) an index of $h_{\mu\nu}$ we find

(2) $$\begin{aligned} \epsilon h^{\alpha}{}_{\nu} &= g^{\alpha\mu}(\epsilon h_{\mu\nu}) = \eta^{\alpha\mu}(\epsilon h_{\mu\nu}) + \epsilon^2 h^{\alpha\mu} h_{\mu\nu} \\ &= \eta^{\alpha\mu}\epsilon h_{\mu\nu} + O(\epsilon^2) \end{aligned}$$

Thus to first order in ϵ we may raise (or lower) indices by using the Galilean metric $\eta_{\mu\nu}$. Similarly contractions may in the same approximation be expressed as

(3) $$\epsilon h \equiv \epsilon h^{\alpha}_{\alpha} = \eta^{\lambda\alpha}\epsilon h_{\lambda\alpha} + O(\epsilon^2)$$

Quite generally, the once contracted Riemann Curvature tensor may be written as

$$(4) \qquad R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\sigma} - \Gamma^{\sigma}_{\mu\sigma,\nu} + \Gamma^{\sigma}_{\mu\nu}\,\Gamma^{\kappa}_{\sigma\kappa} - \Gamma^{\sigma}_{\mu\kappa}\,\Gamma^{\kappa}_{\nu\sigma}$$

where the Christoffel symbols are defined by

$$(5) \qquad \Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}\, g^{\sigma\alpha}\Big[g_{\mu\alpha,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha} \Big]$$

Now since each Christoffel symbol is at least of order ϵ, we may, in our linear approximation, drop products of two or more of them. Accordingly we may write

$$(6) \qquad R_{\mu\nu} = \Gamma^{\sigma}_{\mu\nu,\sigma} - \Gamma^{\sigma}_{\mu\sigma,\nu}$$

In order to evaluate the once contracted curvature tensor, we must first compute the Christoffel symbols to first order in ϵ, and then their appropriate derivatives. Thus we find,

$$(7) \qquad \Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}\left(\eta^{\sigma\alpha} + \epsilon h^{\sigma\alpha}\right)\Big[(\eta_{\mu\alpha} + \epsilon h_{\mu\alpha})_{,\nu} + (\eta_{\alpha\nu,\mu} + \epsilon h_{\alpha\nu,\mu}) - (\eta_{\mu\nu,\alpha} + \epsilon h_{\mu\nu,\alpha}) \Big]$$

Using the fact that derivatives of $\eta_{\mu\nu}$ vanish (since $\eta_{\mu\nu}$ is a constant metric) and keeping only first order terms we have

$$\Gamma^{\sigma}_{\mu\nu} = \epsilon \, \tfrac{1}{2} \eta^{\sigma\alpha} \left[h_{\mu\alpha,\nu} + h_{\alpha\nu,\mu} - h_{\mu\nu,\alpha} \right] + O(\epsilon^2)$$

Hence differentiating

$$\Gamma^{\sigma}_{\mu\nu,\sigma} = \frac{\epsilon}{2} \eta^{\sigma\alpha} \left[h_{\mu\alpha,\nu,\sigma} + h_{\alpha\nu,\mu,\sigma} - h_{\mu\nu,\alpha,\sigma} \right] \tag{8}$$

Similarly,

$$\Gamma^{\sigma}_{\mu\sigma} = \frac{1}{2} g^{\sigma\alpha} \left[g_{\mu\alpha,\sigma} + g_{\alpha\sigma,\mu} - g_{\mu\sigma,\alpha} \right]$$

$$= \frac{1}{2} \left[g^{\sigma\alpha} g_{\mu\alpha,\sigma} + g^{\sigma\alpha} g_{\alpha\sigma,\mu} - g^{\sigma\alpha} g_{\mu\sigma,\alpha} \right]$$

Because of the symmetry $g^{\sigma\alpha} = g^{\alpha\sigma}$ we can write the last term as

$$- g^{\alpha\sigma} g_{\mu\sigma,\alpha}$$

and upon interchanging the roles of the dummy indices α and σ this becomes identical to the first term and thus cancels it. Thus

$$\Gamma^{\sigma}_{\mu\sigma} = \frac{1}{2} g^{\sigma\alpha} \, g_{\alpha\sigma,\mu} = \frac{1}{2} \epsilon \, \eta^{\sigma\alpha} h_{\alpha\sigma,\mu}$$

Differentiating with respect to the index ν

$$\Gamma^{\sigma}_{\mu\sigma,\nu} = \tfrac{1}{2}\epsilon\,\eta^{\sigma\alpha}\,h_{\alpha\sigma,\mu,\nu} = \tfrac{1}{2}\epsilon\,(h^{\alpha}{}_{\alpha})_{,\mu,\nu} = \tfrac{\epsilon}{2}\,h_{,\mu,\nu} \tag{9}$$

Substituting (8) and (9) into the expression for $R_{\mu\nu}$ (6) we have

$$R_{\mu\nu} = \tfrac{\epsilon}{2}\left[-\eta^{\sigma\alpha}h_{\mu\nu,\alpha,\sigma} + h^{\sigma}{}_{\nu,\mu,\sigma} + h^{\sigma}{}_{\mu,\nu,\sigma} - h_{,\mu,\nu}\right] \tag{10}$$

We can look at this approximation procedure from another point of view: We are always free to transform away the gravitational field at a given point P. (a point for which $\Gamma^{\mu}_{\sigma\alpha} = 0$); having done this we can always look in an infinitesimal neighborhood of this point. In this neighborhood the metric will differ only infinitesimally from the Galilean values at the point under consideration. Thus by saying that the difference $g_{\alpha\beta} - \eta_{\alpha\beta}$ be small (of order ϵ) we have not specified the coordinate system uniquely since any point in the infinitesimal neighborhood of P will satisfy this condition and since each point in this neighborhood defines a distinct reference frame. In considering the above remarks it is useful to bear in mind the analogous situation of a sphere (curved surface) in 3 dimensions. At a given point P on a sphere we may always draw a tangent plane (flat surface) and in an infinitesimal neighborhood of this point P the tangent plane (flat surface) will differ infinitesimally from the corresponding point on the surface of the sphere.

The upshot of all this is that since we haven't specified our coordinate system uniquely we can (and indeed must) impose four conditions on $h_{\alpha\beta}$ which of course must be compatible with its being small. We impose the following conditions

$$\psi^{\alpha}{}_{\beta|\alpha} = 0 \qquad \text{where} \qquad \psi^{\alpha}{}_{\beta} \equiv h^{\alpha}{}_{\beta} - \tfrac{1}{2}\,\delta^{\alpha}{}_{\beta}\,h \tag{11}$$

Yet it is still possible that we have not uniquely specified our coordinate system so we shall look at an arbitrary infinitesimal coordinate transformation and see whether the restriction (11) does or does not leave us any freedom. Consider the transformation

$$x^{\alpha} \rightarrow \bar{x}^{\alpha} = x^{\alpha} + \phi^{\alpha} \tag{12}$$

where ϕ^{α} is small (have dropped the smallness parameter for convenience).

Under this transformation the metric transforms as

$$\bar{g}_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial \bar{x}^{\alpha}} \frac{\partial x^{\nu}}{\partial \bar{x}^{\beta}} g_{\mu\nu} = (\phi^{\mu}{}_{,\alpha} - \delta^{\mu}{}_{\alpha})(\phi^{\nu}{}_{,\beta} - \delta^{\nu}{}_{\beta})$$

$$= \delta^{\mu}{}_{\alpha}\, \delta^{\nu}{}_{\beta}\, g_{\mu\nu} - \delta^{\mu}{}_{\alpha}\, \phi^{\nu}{}_{,\beta}\, g_{\mu\nu} - \delta^{\nu}{}_{\beta}\, \phi^{\mu}{}_{,\alpha}\, g_{\mu\nu} + O(\epsilon^2) \tag{13}$$

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta} - \phi_{\alpha,\beta} - \phi_{\beta,\alpha}$$

or

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \phi_{\alpha,\beta} - \phi_{\beta,\alpha} \tag{14}$$

Raising the index with $\eta^{\alpha\delta}$

$$\bar{h}^{\delta}{}_{\beta} = h^{\delta}{}_{\beta} - \phi^{\delta}{}_{,\beta} - \phi_{\beta}{}^{,\delta}$$

and upon contracting δ and β

$$\bar{h} = h - 2\phi^{\lambda}{}_{,\lambda} \tag{15}$$

Now if our coordinate condition is to be preserved under such an infinitesimal transformation, then we must have

$$\bar{h}^{\alpha}{}_{\beta,\alpha} - \tfrac{1}{2}\delta^{\alpha}{}_{\beta}\, \bar{h}_{,\alpha} = h^{\alpha}{}_{\beta,\alpha} - \tfrac{1}{2}\delta^{\alpha}{}_{\beta}\, h_{,\alpha} - \phi^{\alpha}{}_{,\beta,\alpha} - \phi_{\beta}{}^{,\kappa}{}_{,\alpha} + \delta^{\alpha}{}_{\beta}\, \phi^{\lambda}{}_{,\lambda,\alpha} \tag{16}$$

Thus assuming that the coordinate condition is invariant to such a transformation (i.e., holds in both systems) we have

$$0 = -\cancel{\phi^{\alpha}{}_{,\beta,\alpha}} - \phi_{\beta}{}^{,\alpha}{}_{,\alpha} + \cancel{\phi^{\lambda}{}_{,\lambda,\beta}}$$

or finally

$$\Box \phi_{\beta} = 0 \tag{17}$$

Thus our coordinate specification is invariant with respect to a group of infinitesimal transformations generated by any vector solution of the wave equation. Thus we have found that the persistence of our coordinate condition under an infinitesimal transformation leaves us yet some freedom in a choice of coordinate system and hence does not uniquely specify it. We shall look further into the physical meaning of this coordinate condition and the freedom which it allows in the next section. For now we will show that our coordinate condition implies that the <u>last three terms</u> in the expression for $R_{\mu\nu}$ vanish, and that the linearized Einstein equations result in a wave equation with the energy momentum tensor as its source. The last three terms in $R_{\mu\nu}$ Eqn. (10) are

$$h^{\sigma}{}_{\mu,\nu\sigma} + h^{\sigma}{}_{\nu,\mu\sigma} - h_{,\mu,\nu}$$

Using the coordinate conditions $h^{\sigma}{}_{\mu,\sigma} = \frac{1}{2}\,\delta^{\sigma}{}_{\mu}\,h_{,\sigma}$

the above becomes

$$+\frac{1}{2}\delta^{\sigma}{}_{\mu}\, h_{,\sigma,\nu} + \frac{1}{2}\delta^{\sigma}{}_{\nu}\, h_{,\sigma,\mu} - h_{,\mu,\nu}$$

$$= +\frac{1}{2}\, h_{,\mu,\nu} + \frac{1}{2} h_{,\nu,\mu} - h_{,\mu,\nu} \equiv 0$$

Thus $R_{\mu\nu}$ becomes

$$R_{\mu\nu} = -\frac{1}{2}\eta^{\sigma\alpha} h_{\mu\nu,\sigma,\alpha} = -\frac{1}{2}\Box h_{\mu\nu}$$

or

$$R^{\lambda}{}_{\nu} = -\frac{1}{2}\eta^{\lambda\mu}\eta^{\sigma\alpha} h_{\mu\nu,\sigma,\alpha}$$

and

$$R^{\lambda}{}_{\lambda} = R = -\frac{1}{2}\eta^{\lambda\mu}\eta^{\sigma\alpha} h_{\mu\lambda,\sigma,\alpha} = -\frac{1}{2}\eta^{\sigma\alpha}(h^{\lambda}{}_{\lambda})_{,\sigma,\alpha}$$
$$= -\frac{1}{2}\Box h$$

Thus

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{1}{2}\Box\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h\right)$$

Finally Einstein's field equations become

$$-\frac{1}{2}\Box\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h\right) = \frac{8\pi G}{c^2} T_{\mu\nu}$$

$$\Box\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h\right) = -\frac{16\pi G}{c^2} T_{\mu\nu} \qquad (18)$$

2. The Physical Meaning of Coordinate Conditions - Weyl Solutions

We know in general that Einstein's equations are essentially six equations for ten unknowns and that this presents no conceptual problem inasmuch as one could not expect the equations of physics to specify the coordinate system. In other words, since the field equations are specifically made covariant, that is, independent of our choice of coordinates, we could hardly expect the equations to tell us which coordinates to use. Thus we are allowed, and indeed forced to impose four arbitrary conditions upon the field equations and thus pick our coordinate system. This is very analogous to the situation in Electrodynamics which occurs when solving Maxwell's field equations. For in electrodynamics the $\vec{E}$ and $\vec{H}$ (or $\vec{A}$ and ϕ) fields depend upon our frame of reference (in the same way that $g_{\alpha\beta}$ does above). What one does is to pick a frame for which there is a definite relation between $\vec{E}$ and $\vec{H}$ (or $\vec{A}$ and ϕ) i.e., choosing a guage (coordinate conditions) and one finds again an arbitrariness in that there exists a group of transformations which leave the $\vec{E}$ and $\vec{H}$ fields in the same relation to one another; V13,

$$\begin{aligned} \vec{A} &\rightarrow \vec{A}' = \vec{A} + \nabla\Lambda \\ \phi &\rightarrow \phi' = \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t} \end{aligned} \tag{19}$$

Now if A, ϕ are such that they satisfy the guage condition $\vec{\nabla}\cdot\vec{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}=0$ we see that the persistence of this condition for an arbitrary infinitesimal transformation requires that

$$\vec{\nabla}\cdot\vec{A}'+\frac{1}{c}\frac{\partial\phi'}{\partial t}=\underbrace{\vec{\nabla}\cdot\vec{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}}_{=0}+\nabla^2\Lambda-\frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}=0$$

or

$$\Box\Lambda=0 \tag{20}$$

and we see that we have the same additional freedom in our choice of guage as we did in our choice of coordinate conditions. That is, by picking the Lorentz guage $\vec{\nabla}\cdot\vec{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}$ there still remains an arbitrariness to within a ~~secular~~ scalar solution of the wave equation (in gravitational case there was a vector solution of the wave equation). Thus we see that there is a direct analogy between the electromagnetic and gravitational wave equations.

Now the most obvious choice of coordinate conditions are those for which the Einstein equations take on their simplest mathematical form, but this is not of necessity the most physically meaningful form. The remainder of this section will be devoted to showing that in the case we are considering, the inhomogeneous wave equation obtained by using our particular coordinate conditions is indeed the most physical choice. In what follows we consider only the free field equations

$$R_{\mu\nu} = 0 \tag{21}$$

Before we imposed our coordinate conditions the Einstein tensor had the form

$$\begin{aligned} R_{\mu\nu} &= \frac{\epsilon}{2}\left[h^{\sigma}{}_{\mu,\nu,\sigma} + h^{\sigma}{}_{\nu,\mu,\sigma} - \eta^{\sigma\alpha} h_{\mu\nu,\alpha\sigma} - h_{,\mu,\nu} \right] \\ &= \frac{\epsilon}{2}\left[-\Box h_{\mu\nu} + h^{\sigma}{}_{\mu,\sigma,\nu} + h^{\sigma}{}_{\nu,\sigma,\mu} - h_{,\mu,\nu} \right] \\ &= \frac{\epsilon}{2}\left[-\Box h_{\mu\nu} + \left(h^{\sigma}{}_{\mu,\sigma} - \tfrac{1}{2} h_{,\mu}\right)_{,\nu} + \left(h^{\sigma}{}_{\nu,\sigma} - \tfrac{1}{2} h_{,\nu}\right)_{,\mu} \right] \end{aligned} \tag{22}$$

Define

$$\tau_{\nu} = \left(h^{\sigma}{}_{\nu,\sigma} - \tfrac{1}{2} h_{,\nu} \right) \tag{23}$$

so that the field equations $R_{\mu\nu} = 0$ can be written

$$\Box h_{\mu\nu} = \tau_{\mu,\nu} + \tau_{\nu,\mu} \tag{24}$$

Now let τ_{ν} be the source for a wave equation involving an arbitrary function

$$\Box \phi_{\nu} = \tau_{\nu} \tag{25}$$

Substituting this back into (24) we obtain

$$\Box h_{\mu\nu} = \Box \phi_{\mu,\nu} + \Box \phi_{\nu,\mu} \tag{26}$$

This last equation suggests that a possible solution is

$$h_{\mu\nu} = \phi_{\mu,\nu} + \phi_{\nu,\mu} \tag{27}$$

where it must be emphasized that ϕ_μ is arbitrary. To see if this is actually a possible solution we must substitute this back into the wave quation $R_{\mu\nu} = 0$ i.e.,

$$-\Box h_{\mu\nu} + h^{\sigma}{}_{\mu,\sigma,\nu} + h^{\sigma}{}_{\nu,\sigma,\mu} - h_{,\mu,\nu} = 0$$

or

$$-\Box h_{\mu\nu} + \underline{\phi^{\sigma}{}_{,\mu,\sigma,\nu}} + \phi_{\mu}{}^{,\sigma}{}_{,\sigma,\nu} + \underline{\phi^{\sigma}{}_{,\nu,\sigma,\mu}} + \phi_{\nu}{}^{,\sigma}{}_{,\sigma,\mu} - \underline{2\phi_{\lambda}{}^{,\lambda}{}_{,\mu,\nu}} = 0$$

The three underlined terms cancel one another and we are left with

$$-\Box h_{\mu\nu} + \phi_{\mu}{}^{,\sigma}{}_{,\sigma,\nu} + \phi_{\nu}{}^{,\sigma}{}_{,\sigma,\mu} = 0$$

$$-\Box h_{\mu\nu} + \phi_{\mu,\nu}{}^{,\sigma}{}_{,\sigma} + \phi_{\nu,\mu}{}^{,\sigma}{}_{,\sigma} = 0$$

$$-\Box h_{\mu\nu} + \underbrace{\Box(\phi_{\mu,\nu} + \phi_{\nu,\mu})} = 0$$

but by (26) the braced term is exactly $\Box h_{\mu\nu}$ so that $h_{\mu\nu} = \phi_{\mu,\nu} + \phi_{\nu,\mu}$ is a solution. This general class of solutions generated by the four arbitrary functions ϕ_μ are called Weyl solutions and we denote them by a superscript (ω). Now if we take any given solution of the linear equations $h_{\mu\nu}$ we can always construct the functions $\tau_\mu = h^\sigma{}_{\mu,\sigma} - \frac{1}{2} h_{,\mu}$ and can associate functions ϕ_μ with the τ_μ via $\Box\phi_\mu = \tau_\mu$. Thus we have constructed new functions ϕ_μ from a given solution to the linear field equations which can in turn be used to generate a new solution of the field equations, viz.

$$h^{(\omega)}_{\mu\nu} = \phi_{\mu,\nu} + \phi_{\nu,\mu} \tag{28}$$

the associated Weyl solutions.

Since we are dealing with linear equations, the sum (or the difference) of any two solutions is again a solution; thus we have another solution :

$$\hat{h}_{\mu\nu} = h_{\mu\nu} - h^{(\omega)}_{\mu\nu} \tag{29}$$

Thus given a solution to the linearized field equations one can construct uniquely an associated Weyl solution and their difference is also a unique solution.

Note that the given solution $h_{\mu\nu}$ satisfies Eqn. (24)

and that the associated Weyl solution satisfies Eqn. (26). Thus their difference satisfies

$$\Box \hat{h}_{\mu\nu} = 0 \tag{30}$$

or formally we could write it as

$$\Box \hat{h}_{\mu\nu} = \hat{\tau}_{\mu,\nu} + \hat{\tau}_{\nu,\mu} \equiv 0 \tag{31}$$

Only this difference solution will turn out to have physical significance. Note that it satisfies a homogeneous wave equation and propagates with the velocity of light C. The reason for writing Eqn. (31) down is that formally there is always a function $\hat{\tau}_{\mu}$ associated with $\hat{h}_{\mu\nu}$, and in addition one can show that if this function $\hat{\tau}_{\mu}$ satisfies the condition imposed upon it by (31) and if it is also regular everywhere and vanishes at infinity in an asymptotically pseudoeuclidean space then $\hat{\tau}_{\mu} = 0$. Thus Eqn (31) reduces to the system of equations

$$\left\{ \begin{array}{l} \Box \hat{h}_{\mu\nu} = 0 \\ \hat{\tau}_{\mu} = 0 \end{array} \right. \tag{32}$$

which is just the homogeneous wave equation and the associated coordinate condition Eqn. (11). Thus we can identify

$$0 = \hat{\tau}_{\mu} = \psi^{\alpha}{}_{\mu,\alpha} = h^{\alpha}{}_{\mu,\alpha} - \frac{1}{2} h_{,\mu} \tag{33}$$

Now we are in a position to justify why we have said that only the difference solution, i.e., $\hat{h}_{\mu\nu} = h_{\mu\nu} - h^{(\omega)}_{\mu\nu}$ is physically significant. In order to do this we must first understand what the physically significant quantity is. Since the wave phenomena we are considering is that of a metric wave, and since the metric determines the structure of the space it is natural to look at the full Riemann Tensor which determines whether or not the space is flat. (Recall that the vanishing of the full Riemann curvature tensor is a necessary and sufficient condition for the space to be flat.) If the space is flat there is no sense in talking about gravitational waves since gravitation is ascribed the property of curving the space. The full Riemann curvature tensor is

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\delta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\alpha}_{\lambda\gamma}\Gamma^{\lambda}_{\beta\delta} - \Gamma^{\alpha}_{\lambda\delta}\Gamma^{\lambda}_{\beta\gamma} \tag{34}$$

In the linear approximation products of two or more Christoffel symbols may be dropped and we have

$$R^{\alpha}{}_{\beta\gamma\delta} = \frac{\epsilon}{2}\left\{\eta^{\alpha\lambda}\left[h_{\lambda\delta,\beta} + h_{\beta\lambda,\delta} - h_{\beta\delta,\lambda}\right]_{,\gamma} - \eta^{\alpha\lambda}\left[h_{\lambda\gamma,\beta} + h_{\beta\lambda,\gamma} - h_{\beta\gamma,\lambda}\right]_{,\delta}\right\}$$

$$= \frac{\epsilon}{2}\left\{\eta^{\alpha\lambda}\left[h_{\lambda\delta,\beta,\gamma} + h_{\beta\gamma,\lambda,\delta} - h_{\lambda\gamma,\beta,\delta} - h_{\beta\delta,\lambda,\gamma}\right]\right\} \qquad (35)$$

A Weyl solution has the form

$$h^{(\omega)}_{\alpha\beta} = \phi_{\alpha,\beta} + \phi_{\beta,\alpha} \qquad (36)$$

Thus substituting this into the above

$$\begin{aligned} R^{\alpha}{}_{\beta\gamma\delta} = \frac{\epsilon}{2}\Big\{ & \phi^{\alpha}{}_{,\delta,\beta,\gamma} + \phi_{\delta}{}^{,\alpha}{}_{,\beta,\gamma} \\ & + \phi_{\beta,\gamma}{}^{,\alpha}{}_{,\delta} + \phi_{\gamma,\beta}{}^{,\alpha}{}_{,\delta} \\ & - \phi^{\alpha}{}_{,\gamma,\beta,\delta} - \phi_{\alpha}{}^{,\gamma}{}_{,\beta,\delta} \\ & - \phi_{\beta,\delta}{}^{,\alpha}{}_{,\gamma} - \phi_{\delta,\beta}{}^{,\alpha}{}_{,\gamma} \Big\} \equiv 0 \end{aligned} \qquad (37)$$

Thus the Weyl solution corresponds to a null Riemann tensor (in first order) and thus to a <u>flat space</u>. Therefore the Riemann tensor depends only on $h_{\alpha\beta}$. The Weyl solution seems to be purely formal and indeed can be transformed away by proper choice of coordinate system. Recall that upon making a transformation of the type (11) $\bar{X}^{\alpha} = x^{\alpha} + \phi^{\alpha}$

we found that the transformed metric $\bar{h}_{\alpha\beta}$ differed from the original metric $h_{\alpha\beta}$ by a Weyl type solution $\phi_{\alpha,\beta} + \phi_{\beta,\alpha}$ (see Eqn. 14). Hence the influence of the arbitrariness of the coordinate system due to the freedom in choosing the (guage) function ϕ_α is to add a Weyl-type solution to $h_{\alpha\beta}$ and conversely one can eliminate the Weyl part of a solution $h_{\alpha\beta}$ by a simple coordinate transformation. In fact if $h_{\alpha\beta} = h^{(\omega)}_{\alpha\beta}$ (i.e., purely Weyl-type solution), one can easily go to a system where $h_{\alpha\beta} = \eta_{\alpha\beta}$ i.e., a pseudo-Eucledian space. This is in accord with the vanishing of the Riemann tensor for the Weyl solution and and its corresponding to flat space and thus no gravitational field. Thus we have shown that Eqns. (32) represent the meaningful solutions. Note that the condition $\hat{t}_\alpha = 0$ introduces relations among the $\hat{h}_{\alpha\beta}$ in the same way that for instance the Coulomb guage $\vec{\nabla}\cdot\vec{A} = 0$ introduces relations among the components of the vector potential in electrodynamics, e.g., $\vec{k}\cdot\vec{A} = 0$ i.e., polarization effects; a similar "polarization" effect occurs in gravitational waves due to the coordinate condition on $\hat{h}_{\alpha\beta}$.

Cosmology - Static Models

It is our aim in this section to apply the equations of general relativity to the universe as a whole and to use these results together with other assumptions to construct models of our universe. Then, by comparing the predictions of these models with observations, we will be able to determine which of these models, if any, actually corresponds to our universe.

In the discussion of static cosmology the most important observational facts are the following:

1) density - the observable matter in the universe gives $= 7 \times 10^{-31}$ gm/cm^3 ,

2) red shift - distance relation - Hubble has found that a linear relation exists between the observed red shift of a source and the distance to the source.

We will show in this section that none of the static models are able to predict these observational results.

In obtaining a cosmological line element we shall assume that the universe is homogeneous and isotropic and neglect local irregularities in the gravitational field. Hence we take the line element in the general spherically symmetrical static form

$$ds^2 = -e^{\lambda} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 + e^{\nu} dt^2$$

with λ and ν functions of r alone. Also if we assume that the universe is filled with a perfect fluid with pressure p_o and density ρ_{oo}, the field equations become

$$8\pi p_o = e^{-\lambda}\left[\frac{\nu'}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2} + \Lambda$$

$$8\pi \rho_{oo} = e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{1}{r^2}\right] + \frac{1}{r^2} - \Lambda$$

$$\frac{dp_o}{dr} = -\frac{\rho_{oo} + p_o}{2}\nu'$$

where the primes refer to differentiation with respect to r and Λ is the cosmological constant. (See Tolman, Relativity, Thermodynamics and Cosmology, p. 242.)

We can now obtain the only possibilities for a static homogeneous and isotropic model, by imposing the following conditions on the above equations. First that the pressure as measured by a local observer shall be the same everywhere, because of the assumed homogeneity of the model; secondly, that the proper density ρ_{oo} shall everywhere be the same, again owing to the homogeneity of the model; and thirdly that the line element shall reduce to special relativity for small values of r, $\lambda = \nu = 0$, owing to the known validity of the special theory of relativity for a limited space-time region.

Since P_0 must be independent of space coordinates, we have

$$\frac{dP_0}{dr} = 0$$

or

$$\frac{\rho_{00} + P_0}{2} \nu' = 0$$

This leads to three possibilities

$$\nu' = 0$$

$$\rho_{00} + P_0 = 0$$

or $\nu' = 0$ and $\rho_{00} + P_0 = 0$

I. <u>The Einstein Universe</u>

1) The line element.

The Einstein line element arises from choosing

$$\nu' = 0$$

Integrating this equation, and keeping condition three in mind ($\nu \to 0$ as $r \to 0$), we obtain

$$\nu = 0$$

Then

$$8\pi P_o = \frac{e^{-\lambda}}{r^2} - \frac{1}{r^2} + \Lambda ,$$

or

$$e^{-\lambda} = 1 - (\Lambda - 8\pi P_o) r^2 .$$

If we define R as

$$(\Lambda - 8\pi P_o) = 1/R^2 ,$$

the Einstein line element can be written as

$$ds^2 = \frac{-dr^2}{1 - r^2/R^2} - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 + dt^2$$

2) The geometry.

With the line element given above we can find the volume of the Einstein universe to be

$$V = \int_0^R \int_0^\pi \int_0^{2\pi} \frac{r^2 \sin\theta \, dr \, d\theta \, d\phi}{\sqrt{1 - r^2/R}}$$

$$= 4\pi \int_0^R \frac{r^2 \, dr}{\sqrt{1 - r^2/R}}$$

$$= \pi^2 R^3 .$$

The spatial geometry defined by this volume and the line element

$$ds^2 = - \frac{dr^2}{1 - r^2/R^2} - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 + dt^2 ,$$

is called elliptical.

The geometry can be interprted somewhat differently if we introduce new coordinates. Consider the transformation

$$z_1 = R \sqrt{1 - r^2/R^2}$$

$$z_2 = r \cos\theta \qquad z_3 = r \sin\theta \cos\phi$$

$$z_4 = r \sin\theta \sin\phi$$

where

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = R^2$$

In terms of these coordinates the line element has the form

$$ds^2 = - (dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2) + dt^2$$

and we may consider the spatial part of the Einstein universe as the three-dimensional spherical surface $z_1^2 + z_2^2 + z_3^2 + z_4^2 = R^2$ embedded in the four-dimensional Euclidean space (z_1, z_2, z_3, z_4). Introducing polar coordinates on the sphere

$$z_1 = R\cos\psi \qquad z_2 = R\sin\psi\cos\theta$$

$$z_3 = R\sin\psi\sin\theta\cos\phi \qquad z_4 = R\sin\psi\sin\theta\sin\phi$$

we have

$$ds^2 = -R^2(d\psi^2 + \sin^2\psi\, d\theta^2 + \sin^2\psi\sin^2\theta\, d\phi^2) + dt^2.$$

Each point on the sphere corresponds to one set of values (ψ, θ, ϕ), in the intervals

$$0 \leq \psi \leq \pi,$$

$$0 \leq \theta \leq \pi, \qquad 0 \leq \phi \leq 2\pi$$

The volume of the Einstein universe in terms of these new coordinates is

$$V = \int_0^\pi d\psi \int_0^\pi d\theta \int_0^{2\pi} d\phi \; R^3 \sin^2\psi \sin\theta$$

$$= 4\pi R^3 \int_0^\pi \sin^2\psi\, d\psi$$

$$= 2\pi^2 R^3 .$$

Thus the volume in this so-called spherical space is twice that in the elliptical space. This comes from the fact that

$$r = R \sin \psi$$

which shows that the elliptical space covers only the hemisphere $0 \leq \psi \leq \pi/2$. In the elliptical space antipodal points on the sphere are counted as one point.

We can easily calculate the distance around the universe in terms of the Z's. If we consider motion only in the Z_1-Z_2 plane, we have

$$z_1^2 + z_2^2 = R^2 ,$$

$$z_1 \, dz_1 = - z_2 \, dz_2 ,$$

and

$$ds^2 = dz_1^2 + dz_2^2$$

The element of arc length can be reduced to

$$ds = \frac{R \, dz_2}{\sqrt{R^2 - z_2^2}} ,$$

and then the distance around the universe is given by

$$\ell = 4 \int_0^R ds$$

$$= 4R \int_0^R \frac{dx}{\sqrt{R^2 - x^2}}$$

$$= 4R \sin^{-1} x/R \Big|_0^R$$

$$= 2\pi R$$

The distance around the universe in the case of elliptical geometry will be half this value.

3) Density and Pressure.

We have found that

$$e^{-\lambda} = 1 - r^2/R^2 ,$$

where

$$(\Lambda - 8\pi\rho_0) = 1/R^2 .$$

Introducing this into the equation

$$8\pi\rho_{00} = e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{1}{r^2}\right] + \frac{1}{r^2} - \Lambda$$

we obtain

$$8\pi \rho_{00} = \frac{2r}{rR^2} - \frac{1}{r^2} + \frac{1}{R^2} + \frac{1}{r^2} - \Lambda$$

$$= 3/R^2 - \Lambda .$$

If we solve for R and Λ in terms of ρ_{00} and P_0 , we find

$$\Lambda = 4\pi \left(\rho_{00} + 3 P_0 \right)$$

$$1/R^2 = 4\pi \left(\rho_{00} + P_0 \right) .$$

Therefore, since ρ_{00} and P_0 are positive quantities, we conclude that Λ and R^2 are both positive. We regard Λ and R^2 as adjustable parameters which depend on the nature of the fluid used to fill our model.

If we consider our model as filled with matter exerting no pressure, we have

$$4\pi \rho_{00} = \Lambda = 1/R^2 ,$$

which for $\rho_{00} \sim 10^{-31}$ gm/cm^3 gives $\Lambda \sim 10^{-58}$ cm^{-2}. On the other hand if we took the model as filled with radiation

$$\rho_{00} = 3 P_0) ,$$

$$\Lambda = 3/2R^2 ,$$

and

$$4\pi P_0 = 1/4R^2 \qquad 4\pi \rho_{00} = 3/4R^2$$

Finally, if we take

$$\rho_{00} = p_0 = 0$$

we have

$$\Lambda = 1/R^2 = 0$$

and the Einstein universe would degenerate into the flat space time of special relativity.

It should be noted that, in order for the Einstein universe to contain any matter, it is necessary that Λ be non-zero and R be positive corresponding to a spatially closed universe of finite spatial volume.

4) Particles and light rays.

The motion of a free particle in the gravitational field corresponding to the Einstein universe is given by the geodesic equation

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0$$

where the Γ's are determined from the line element. Since we are considering a static model, we would hope that the particles of the model would be at rest with respect to the spatial coordinates. If we consider particles with zero spatial velocity,

$$\frac{dr}{ds} = \frac{d\theta}{ds} = \frac{d\phi}{ds} = 0 \; ,$$

the equations of motion reduce to

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{44} \left(\frac{dt}{ds}\right)^2 = 0 \; .$$

Upon calculating the Christoffel symbols and substituting them into the above expression we find that

$$\frac{d^2r}{ds^2} = \frac{d^2\theta}{ds^2} = \frac{d^2\phi}{ds^2} = 0 \; .$$

Thus particles at rest would remain at rest and the Einstein model could be expected to persist in the assumed static state.

The velocity of light can be determined by setting $ds = 0$, for motion in the radial direction we have

$$\frac{dr}{dt} = \pm \sqrt{1 - r^2/R^2}$$

In order to calculate the time it would take for a light signal to travel around the universe, we use the coordinates (z_1, z_2, z_3, z_4) introduced above and find

$$dt^2 = dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2$$

If we adjust the coordinates so that the motion only takes place in the Z_1-Z_2 plane, we have

$$z_1^2 + z_2^2 = R^2,$$

$$z_1 \, dz_1 = - z_2 \, dz_2$$

and

$$dt = \sqrt{dz_1^2 + dz_2^2}$$

Then

$$t = 4 \int_0^R \sqrt{\frac{z_2^2}{z_1^2} + 1} \; dz_2$$

$$= 4R \int_0^R \frac{dz_2}{\sqrt{R^2 - z_2^2}}$$

$$= 4R \sin^{-1} \frac{z_2}{R} \Big|_0^R$$

$$= 2\pi R .$$

We now investigate the possibility of a red shift in the Einstein model. Consider an observer at the origin of coordinates

$r = 0$ abd a siyrce at $r = r$, then a light signal leaving the source at a time t_1 will arrive at the observer at t_2,

$$t_2 = t_1 + \int_0^r \frac{dr}{\sqrt{1 - r^2/R^2}}$$

$$= t_1 + R \sin^{-1}(r/R)$$

Hence, since r is constant, the internal δt_2 between the reception of two successive wave crests would be equal to the internal δt_1 between their emission

$$\delta t_2 = \delta t_1$$

But according to the line element, the quantity t is the proper time for both the source and the observer, and therefore, since the period of the light would be the same at $r = 0$ and $r = r$, there is no red shift. We can conclude that there would be no systematic connection between observed wave length and the distance from the observer to the source. There could however be small Doppler effects due to the individual motions of the sources.

II. The de Sitter Universe

1) The line element

The de Sitter universe is characterized by the condition that

$$\rho_{oo} + P_o = 0$$

But, since

$$8\pi(\rho_{oo} + P_o) = e^{-\lambda}\,\frac{\nu' + \lambda'}{r},$$

this condition reduces to

$$\lambda' = -\nu'.$$

The solution which reduces to special relativity at $r = 0$ is

$$\lambda = -\nu.$$

Now consider the equation

$$8\pi\,\rho_{oo} = e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{1}{r^2}\right] + \frac{1}{r^2} - \Lambda$$

or

$$(8\pi\,\rho_{oo} + \Lambda)\,r^2 - 1 = e^{-\lambda}\,r\,\lambda' - e^{-\lambda}$$

The solution is

$$(8\pi\,\rho_{oo} + \Lambda)\,\frac{r^3}{3} - r = -r\,e^{-\lambda} + A$$

or

$$e^{-\lambda} = 1 - \frac{\Lambda + 8\pi\rho_{oo}}{3}\,r^2 + \frac{A}{r}$$

where A is the constant of integration. Again requiring that $e^{-\lambda} \to 1$ as $r \to 0$, we put $A = 0$ and obtain

$$e^{-\lambda} = e^{\nu} = 1 - \frac{\Lambda + 8\pi \rho_{00}}{3} r^2 .$$

If we introduce the constant R such that

$$\frac{\Lambda + 8\pi \rho_{00}}{3} = \frac{1}{R^2} ,$$

the line element becomes

$$ds^2 = -\frac{dr^2}{1 - r^2/R^2} - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2$$

$$+ (1 - r^2/R^2) \, dt^2 .$$

2) The geometry

By the substitution

$$r = R \sin \chi$$

we obtain

$$ds^2 = -R^2 d\chi^2 - R^2 \sin^2\chi \, d\theta^2$$

$$- R^2 \sin^2\chi \sin^2\theta \, d\phi^2 + \cos^2\chi \, dt^2 .$$

Another form may be obtained by using the transformation

$$\alpha = r \sin\theta \cos\phi \qquad \beta = r \sin\theta \sin\phi$$

$$\gamma = r \cos\theta \qquad \delta + \epsilon = R e^{t/R} \left(1 - r^2/R^2\right)^{1/2}$$

$$\delta - \epsilon = R e^{-t/R} \left(1 - r^2/R^2\right)^{1/2} .$$

Then

$$ds^2 = -d\alpha^2 - d\beta^2 - d\gamma^2 - d\delta^2 + d\epsilon^2$$

with

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 - \epsilon^2 = R^2 .$$

This gives a spatially closed model provided R is positive and finite. We can embed the whole of space-time in a five-dimensional Euclidean space by using the transformation

$$z_1 = i\alpha , \quad z_2 = i\beta , \quad z_3 = i\gamma , \quad z_4 = i\delta , \quad z_5 = \epsilon ,$$

which gives

$$ds^2 = dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2 + dz_5^2$$

and

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = (iR)^2$$

Finally, introducing the new variables

$$\bar{r} = \frac{r}{(1 - r^2/R^2)^{1/2}} \, e^{-t/R}$$

and

$$\bar{t} = t + \tfrac{1}{2} R \log \left(1 - r^2/R^2\right),$$

we have

$$ds^2 = - e^{2\bar{t}/R} \left(d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta \, d\phi^2 \right) + d\bar{t}^2 .$$

This can be written as

$$ds^2 = - e^{2kt} \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right) + dt^2$$

where

$$k = 1/R .$$

This is a non-static form of the de Sitter line element and is useful in discussing the relation between red shift and distance. (See Tolman p. 356.)

3) Absence of matter and radiation.

The de Sitter universe is characterized by

$$\rho_{00} + p_0 = 0$$

The proper matter density ρ_{00} is by definition either zero or a positive quantity and p_0 must be non-negative if the model is to be stable against collapse. Thus we have

$$\rho_{00} = 0 \quad \text{and} \quad p_0 = 0$$

or

$$1/R^2 = \Lambda/3$$

Therefore the de Sitter model can be regarded as spatially closed if the cosmological constant is positive, as degenerating into the open flat space-time of special relativity if the cosmological constant is equal to zero, and as spatially open but curved if the cosmological constant should be negative.

4) Particles and light rays.

The motion of test particles and light rays is governed by the equations for a geodesic

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0$$

Substituting the values for $\Gamma^\alpha_{\beta\gamma}$ as determined from the line element we find

$$\frac{d^2r}{ds^2} + \frac{1}{2} \frac{d\lambda}{dr} \left(\frac{dr}{ds}\right)^2 - r\, e^{-\lambda} \left(\frac{d\theta}{ds}\right)^2$$

$$- r \sin^2\theta\, e^{-\lambda} \left(\frac{d\phi}{ds}\right)^2 + \frac{1}{2} e^{\nu-\lambda} \frac{d\nu}{dr} \left(\frac{dt}{ds}\right)^2 = 0,$$

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0 ,$$

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} + 2\cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} = 0 ,$$

$$\frac{d^2t}{ds^2} + \frac{d\nu}{dr}\frac{dr}{ds}\frac{dt}{ds} = 0.$$

If we choose the coordinates such that the motion of interest is initially in the plane $\theta = \frac{1}{2}\pi$, then according to the second of the above equations the motion will remain permanently in that plane and the equations will reduce to

$$\frac{d^2r}{ds^2} + \frac{1}{2}\frac{d\lambda}{dr}\left(\frac{dr}{ds}\right)^2 - r\,e^{-\lambda}\left(\frac{d\phi}{ds}\right)^2$$

$$+ \frac{1}{2}e^{\nu-\lambda}\frac{d\nu}{dr}\left(\frac{dt}{ds}\right)^2 = 0 ,$$

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} = 0 ,$$

$$\frac{d^2 t}{ds^2} + \frac{d\nu}{ds}\frac{dt}{ds} = 0 .$$

Integrating these equations we obtain

$$e^{\lambda}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2 - e^{\nu}\left(\frac{dt}{ds}\right)^2 + 1 = 0$$

$$\frac{d\phi}{ds} = \frac{h}{r}$$

$$\frac{dt}{ds} = k\, e^{-\nu}$$

where h and k are constants of integration. Substituting for λ and ν according to

$$e^{-\lambda} = e^{\nu} = 1 - r^2/R^2$$

we find that

$$\frac{dr}{ds} = \left(k^2 - 1 + \frac{r^2}{R^2} - \frac{h^2}{r^2} + \frac{h^2}{R^2} \right)^{1/2}$$

$$\frac{d\phi}{ds} = \frac{h}{r^2}$$

$$\frac{dt}{ds} = \frac{k}{1 - r^2/R^2}$$

From the first two of these equations we find the equations for the orbit of a particle

$$\phi = \int_{r_0}^{r} \frac{h\,dr}{r^2\left(r^2/R^2 + k^2 - 1 - h^2/r^2 + h^2/R^2\right)^{1/2}} + \phi_0 .$$

According to Newtonian mechanics the orbit of a particle in a potential given as $V(r)$ as

$$\phi = \int_{r_0}^{r} \frac{dr}{r^2\left(\frac{2mE}{\ell^2} - \frac{2mV(r)}{\ell^2} - 1/r^2\right)^{1/2}} + \phi_0$$

where m is the mass of the particle, E its energy and ℓ the angular momentum. (See Goldstein, Classical Mechanics, p. 73) Thus in the de Sitter model we have that V(r) is proportional to $-r^2$ and therefore the force acting on a particle is proportional to r.

The trajectory of a light ray is characterized by the condition that ds = 0. Since this introduces infinities in the geodesic equation, we write

$$ds = \beta \, d\alpha$$

and consider the limit as $\beta \to 0$. Substituting the above expression into the first integrals of the geodesic equation we find

$$e^{\lambda}\left(\frac{dr}{d\alpha}\right)^2 + r^2\left(\frac{d\phi}{d\alpha}\right)^2 + e^{\nu}\left(\frac{dt}{d\alpha}\right)^2 + \beta = 0$$

$$\frac{d\phi}{d\alpha} = \frac{h\beta}{r^2}$$

$$\frac{dt}{d\alpha} = \beta k e^{-\nu}$$

Now as $\beta \to 0$ we must have that

$$h \to A/\beta$$

and

$$k \to B/\beta$$

where A and B are constants. This insures that

$$\frac{d\phi}{d\alpha} \quad \text{and} \quad \frac{dt}{d\alpha}$$

will remain finite. Solving the above equations for $dr/d\alpha$ we find that

$$\frac{dr}{d\alpha} = \left(\beta^2 k^2 - \frac{h^2\beta^2}{r^2} + \frac{h^2\beta^2}{R^2} - \beta + \frac{\beta r^2}{R} \right)^{1/2}$$

or

$$d\phi = \frac{dr}{r^2 \left(\frac{k^2}{h^2} - \frac{1}{r^2} + \frac{1}{R^2} - \frac{1}{\beta h^2} + \frac{r^2}{\beta h^2 R^2} \right)^{1/2}}$$

If we now consider the limit $\beta \to 0$ ($ds \to 0$) , we have

$$d\phi = \lim_{\beta \to 0} \frac{dr}{r^2 \left(\frac{B^2}{A^2} - \frac{1}{r^2} + \frac{1}{R^2} - \frac{\beta}{A^2} + \frac{\beta r^2}{A^2 R^2} \right)^{1/2}}$$

or

$$d\phi = \frac{dr}{r^2 \left(C - \frac{1}{r^2} + \frac{1}{R^2} \right)^{1/2}}$$

where $C = B^2/A^2$. Redefining the constant C we have

$$\phi = \int \frac{dr}{r\,(c r^2 - 1)^{1/2}} - D$$

$$= \sin^{-1} b/r - D$$

where b is defined as $1/\sqrt{c}$. Thus

$$\sin(\phi + D) = b/r$$

and

$$\sin\phi \cos D + \cos\phi \sin D = b/r ,$$

or

$$r \sin\phi + r \tan D \cos\phi = b/\cos D .$$

Hence light rays travel in straight lines in the de Sitter universe.

Let us now turn to a discussion of the red shift in the de Sitter universe. For the case of purely radial motion the velocity of light can be determined from the line element to be

$$\frac{dr}{dt} = \pm \sqrt{1 - \frac{r^2}{R^2}} .$$

Thus light leaving a particle located at r at time t_1 would arrive at the origin at time t_2 given by

$$t_2 = t_1 + \int_0^r \frac{dr}{1 - r^2/R^2},$$

or

$$t_2 = t_1 + \frac{R}{2} \log \left(\frac{R + r}{R - r} \right).$$

Note that it would take an infinite amount of time for light to travel from $r = R$ to the origin. Thus no information could be obtained from the region $r > R$ and R could be called the distance to the horizon of the universe.

The time interval δt_2 between the reception of two successive wave crests would be related to the time interval between their emission by the equation

$$\delta t_2 = \left[1 + \frac{1}{1 - r^2/R^2} \frac{dr}{dt} \right] \delta t_1$$

where dr/dt is the radial velocity of the particle at the time of emission. The relation between δt_2 and the proper time interval for an observer on the moving particle can be determined from the equation

$$\frac{dt}{ds} = \frac{k}{1 - r^2/R^2}.$$

Thus we have

$$\delta t_1^\circ = \frac{1 - r^2/R^2}{k} \, \delta t_1$$

and for the observer at the origin

$$\delta t_2^\circ = \delta t_2 .$$

If λ is the wave length of the light emitted from the moving particle, the wave length observed at the origin, $\lambda + \delta\lambda$, is given by

$$\frac{\lambda}{\delta t_1^\circ} = \frac{\lambda + \delta\lambda}{\delta t_2^\circ}$$

or

$$\frac{\lambda + \delta\lambda}{\lambda} = \frac{\delta t_2^\circ}{\delta t_1^\circ} = \frac{1}{\dfrac{1 - r^2/R^2}{k}} \, \frac{\delta t_2}{\delta t_1}$$

$$= \frac{1 + \dfrac{1}{1 - r^2/R^2} \dfrac{dr}{dt}}{\dfrac{1 - r^2/R^2}{k}}$$

$$= \frac{k}{1 - r^2/R^2} + \frac{k}{(1 - r^2/R^2)^2} \, \frac{dr}{dt} .$$

Since k must be positive for $r < R$, the de Sitter model allows for both red and violet shifts, but it favors red shifts.

It is possible, by the introduction of an additional hypothesis, to obtain a linear relation between the red shift and distance (see Tolman, p. 356), but the de Sitter model is still unsatisfactory because it corresponds to an empty universe.

III. The line element of special relativity.

If we choose the third of the possibilities which lead to a zero pressure gradient,

$$\nu' = 0 \quad \text{and} \quad \rho_{00} + p_0 = 0,$$

the field equations require that

$$\nu' = -\lambda'$$

This, together with the condition that ν and λ approach one as r goes to zero, leads to the result

$$\lambda = \nu = 0$$

and the line element

$$ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 + dt^2.$$

This line element corresponds to the flat space of special relativity.

NON-STATIC COSMOLOGY

A. Investigation of line elements.

One of course assumes spatial isotropy. One also chooses to consider, for convenience, a co-moving coordinate system. This is a system in which all components of velocity are zero. The most general line element in co-moving coordinates which is spatially isotropic is:

$$ds^2 = -e^{\lambda} dr^2 - e^{\mu}(r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2) + e^{\nu} dt^2 + 2a\,dr\,dt \qquad (1)$$

At this point we would like to simplify this expression without changing the co-moving character of our coordinates. In particular we would like to reduce the unknown functions λ, μ, ν to manageable proportions and eliminate the cross term drdt.

Since our coordinates are co-moving

$$\frac{dr}{ds} = \frac{d\theta}{ds} = \frac{d\phi}{ds} = 0$$

because all components of velocity are zero. Substitute a new time-like variable t'

$$dt' = \eta(a\,dr + e^{\nu} dt)$$

where η is an integrating factor chosen to make the right side a perfect differential. Then

$$e^{\nu} dt^2 + 2a\, dr\, dt = \frac{dt'^2}{\eta^2 e^{\nu}} - \frac{a^2}{e^{\nu}} dr^2$$

Substituting into Eq. (1) and dropping the prime

$$ds^2 = -e^{\lambda} dr^2 - e^{\mu}(r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2) + e^{\nu} dt^2$$

where λ, μ and ν are now functions of r and t. Notice that the coordinates r, θ, ϕ haven't been changed so we still have a co-moving system.

Consider now the components of the gravitational acceleration for a free test particle. These are determined from the geodesic equation. For co-moving coordinates $\left(\frac{dr}{ds} = \frac{d\theta}{ds} = \frac{d\phi}{ds} = 0\right)$ the geodesic equations become

$$\frac{d^2 r}{ds^2} = -\Gamma^{1}_{44} \left(\frac{dt}{ds}\right)^2$$

$$\frac{d^2 \theta}{ds^2} = -\Gamma^{2}_{44} \left(\frac{dt}{ds}\right)^2$$

$$\frac{d^2 \phi}{ds^2} = -\Gamma^{3}_{44} \left(\frac{dt}{ds}\right)^2$$

However, our test particle is at rest with respect to an observer in a co-moving coordinate system. The assumption of spatial isotropy insures the physical results that are observed are independent of direction. Hence these accelerations are all zero.

Hence

$$\Gamma^{1}_{44} = \Gamma^{2}_{44} = \Gamma^{3}_{44} = 0$$

since $\frac{dt}{ds}$ is not in general zero.

Earlier in the notes it has been shown that

$$\Gamma^{\nu}_{\mu\mu} = -\frac{1}{2} g^{\nu\nu} \frac{\partial g_{\mu\mu}}{\partial x^{\nu}}$$

For the line element we have chosen

$$g = \begin{pmatrix} e^{\lambda} & 0 & 0 & 0 \\ 0 & e^{\mu} r^{2} & 0 & 0 \\ 0 & 0 & e^{\mu} r^{2} \sin^{2}\theta & 0 \\ 0 & 0 & 0 & e^{\nu} \end{pmatrix}$$

so

$$\Gamma^{1}_{44} = 0 = -\frac{1}{2} e^{\lambda + \nu} \nu' \Rightarrow \frac{\partial \nu}{\partial r} = 0$$

$$\Gamma^{2}_{44} = 0 = -\frac{1}{2} e^{\mu+\nu} r^2 \frac{\partial \nu}{\partial \theta} \Rightarrow \frac{\partial \nu}{\partial \theta} = 0$$

$$\Gamma^{3}_{44} = 0 = -\frac{1}{2} e^{\mu+\nu} r^2 \sin^2\theta \frac{\partial \nu}{\partial \phi} \Rightarrow \frac{\partial \nu}{\partial \phi} = 0$$

Thus ν can only be a function of t . This knowledge permits us to introduce a new time variable

$$dt' = e^{\frac{1}{2}\nu} dt$$

without altering the co-moving character of the coordinates. Then

$$ds^2 = -e^{\lambda} dr^2 - e^{\mu}(r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2) + dt^2$$

From our assumption of spatial isotropy and our choice of co-moving coordinates we have obtained a separation of space-time into space and a universal time orthogonal to this space.

According to this form of the line element

$$t = t_o$$

is now the proper time as measured by a local observer at rest with respect to matter in his neighborhood. The proper distances along the coordinate axes are:

$$\delta l_1 = e^{\frac{1}{2}\lambda}\,\delta r \qquad \delta l_2 = e^{\frac{1}{2}\mu}\, r\,\delta\Theta \qquad \delta l_3 = e^{\frac{1}{2}\mu}\, r \sin\Theta\,\delta\phi$$

The fractional rate of change of these proper distances with time is

$$\frac{1}{\delta l_1}\frac{\partial(\delta l_1)}{\partial t_0} = \frac{1}{2}\frac{\partial\lambda}{\partial t} \qquad \frac{1}{\delta l_2}\frac{\partial(\delta l_2)}{\partial t} = \frac{1}{2}\frac{\partial\mu}{\partial t} \qquad \frac{1}{\delta l_3}\frac{\partial(\delta l_3)}{\partial t_0} = \frac{1}{2}\frac{\partial\mu}{\partial t}$$

The assumption of spatial isotropy now implies

$$\frac{\partial\lambda}{\partial t} = \frac{\partial\mu}{\partial t}$$

This result indicates a new transformation which can simplify the line element without altering the co-moving character of the coordinates

$$\frac{dr'}{r'} = e^{\frac{1}{2}(\lambda-\mu)}\frac{dr}{r}$$

Dropping primes the line element becomes:

$$ds^2 = -e^{\mu}\left(dr^2 + r^2 d\Theta^2 + r^2 \sin^2\Theta\, d\phi^2\right) + dt^2$$

Consider again the fractional rate of change of the proper distance between neighboring particles of this model

$$\frac{1}{\delta l_0}\frac{\partial(\delta l_0)}{\partial t} = \frac{1}{2}\frac{\partial \mu}{\partial t}$$

From considerations of spatial isotropy this quantity should be independent of r . Otherwise one could study the entire structure of the galaxy from this model.

$$\frac{\partial^2 \mu}{\partial r \partial t} = 0$$

Then μ can be at most the sum of a function of r and t .

$$\mu(r,t) = f(r) + g(t)$$

Using this non-static isotropic line element it can be shown (Relativity, Thermodynamics and Cosmology, Tolman, p. 251-252) that the surviving terms of the energy-momentum tensors are

$$8\pi T_1{}^1 = -e^{-\mu}\left(\frac{f'^2}{4} + \frac{f'}{r}\right) + \ddot{g} + \frac{3}{4}\dot{g}^2 - \Lambda \qquad (1)$$

$$8\pi T_2{}^2 = 8\pi T_3{}^3 = -e^{-\mu}\left(\frac{f''}{2} + \frac{f'}{2r}\right) + \ddot{g} + \frac{3}{4}\dot{g}^2 - \Lambda \qquad (2)$$

$$8\pi T_4{}^4 = -e^{-\mu}\left(f'' + \frac{f'^2}{4} + \frac{2f'}{r}\right) + \frac{3}{4}\dot{g}^2 - \Lambda \qquad (3)$$

where accents denote differentiation with respect to r and dots with respect to t .

From spatial isotropy the measurements of stress should be symmetric with respect to x, y, and z directions. Hence $T_1^{\ 1} = T_2^{\ 2} = T_3^{\ 3}$ and

$$\frac{f'^2}{4} + \frac{f'}{r} = \frac{f''}{2} + \frac{f'}{2r}$$

$$\frac{d^2f}{dr^2} - \frac{1}{2}\left(\frac{df}{dr}\right)^2 - \frac{1}{r}\frac{df}{dr}$$

As a first integral

$$\frac{df}{dr} = C_1 r e^{\frac{1}{2}f}$$

$$\int e^{-\frac{1}{2}f} df = \int C_1 r\, dr$$

$$\frac{e^{-\frac{1}{2}f}}{-1/2} = \frac{1}{2} C_1 r^2 + C_2$$

$$e^{-\frac{1}{2}f} = C_2 - \frac{1}{4} C_1 r^2$$

$$e^f = \frac{1/C_2^2}{\left[1 - \frac{1}{4}\frac{C_1}{C_2} r^2\right]^2}$$

Let $c_1/c_2 = -\frac{1}{R_o^2}$ where R_o is a constant which can be positive, negative, or infinite. Thus the line element becomes:

$$ds^2 = \frac{-e^{g(t)}}{[1 + r^2/4R_o^2]^2}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right) + dt^2$$

This line element has been derived from extremely straightforward and simple assumptions. If later observations lead to contradictions one must modify either the principles of relativistic mechanics or the assumption of spatial isotropy.

The line element can be written in several different forms which aid in understanding the implied geometry.

a) By the familiar transformation

$$x = r \sin\theta \cos\phi \qquad y = r \sin\theta \sin\phi \qquad z = r \cos\theta$$

one gets

$$ds^2 = \frac{-e^{g(t)}}{[1 + r^2/4R_o^2]^2}\left(dx^2 + dy^2 + dz^2\right) + dt^2$$

which emphasizes the spatial isotropy.

b) By substituting

$$\bar{r} = \frac{r}{1 + r^2/4R_o^2}$$

the line element becomes

$$ds^2 = -e^{g(t)}\left[\frac{d\bar{r}^2}{1-\bar{r}^2/R_0^2} + \bar{r}^2 d\theta^2 + \bar{r}^2\sin^2\theta\, d\phi^2\right] + dt^2$$

This shows the relation between the non-static line element and the static Einstein line element.

c) Substituting $\bar{r} = R_0 \sin\chi$

$$ds^2 = -R_0^2 e^{g(t)}\left(d\chi^2 + \sin^2\chi\, d\theta^2 + \sin^2\chi \sin^2\theta\, d\phi^2\right) + dt^2$$

d) By introducing a larger number of dimensions

$$z_1 = R_0\left[1 - \bar{r}^2/R_0^2\right]^{1/2} \qquad z_2 = \bar{r}\sin\theta\cos\phi$$

$$z_3 = \bar{r}\sin\theta\sin\phi \qquad z_4 = \bar{r}\cos\theta$$

where

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = R_0^2$$

$$ds^2 = -e^{g(t)}\left(dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2\right) + dt^2$$

Thus we can imagine our original space as embedded in a Euclidean space of a larger number of dimensions. The spatial extent of this non-static universe at a given time is a three dimensional spherical surface

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = R_0^2$$

embedded in a four-dimensional Euclidean space (Z_1, Z_2, Z_3, Z_4). The proper distance for the spatial coordinates are

$$\delta l_i = e^{\frac{1}{2} g(t)} dz_i$$

Therefore the radius of the spherical surface is

$$R = R_0 e^{\frac{1}{2} g(t)}$$

This quantity is conveniently spoken of as the radius of the non-static universe. Thus the expansion or contraction of g(t) controls the expansion or contraction of the universe. But note that, as defined, the radius R_0 could be real, imaginary or infinite.

If the radius is assumed to be real the universe will be closed. The volume at any time t is given by integrating the line element (c)

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{\pi} R_0^3 e^{\frac{3}{2} g(t)} \sin^2 \chi \sin\theta \, d\chi \, d\theta \, d\phi$$

$$= 2\pi^2 R_0^3 e^{\frac{3}{2} g(t)}$$

The "circumference" around the universe would be $\ell_o = 2\pi R_o e^{1/2\, g(t)}$

On the other hand, if R_o is imaginary or infinite, the model will be spatially open. The volume can be most conveniently calculated by using the line element (b).

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{e^{3/2\, g(t)}}{\left[1 + \bar{r}^2/A^2\right]^{1/2}}\, \bar{r}^2 \sin\theta \; d\bar{r}\, d\theta\, d\phi = \infty$$

B. Density and pressure in a non-static universe.

At this point no assumptions have been made regarding the nature of the matter filling the model except that it obeys Einstein's field equations

$$-8\pi T_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}$$

We now introduce the assumption that the material filling the model constitutes a perfect fluid. Therefore we can use the expression for the energy momentum tensor of a perfect fluid.

$$T^{\mu\nu} = (\rho_{oo} + P_o)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - g^{\mu\nu} P_o$$

where ρ_{00} and P_o are the macroscopic density and pressure as seen by a local observer at rest in the fluid. $\frac{dx^\mu}{ds}$ are the components of the macroscopic velocity of the fluid with respect to the co-moving coordinates.

Using spherical polar coordinates the line element becomes

$$ds^2 = -e^{g(t)} \frac{1}{[1 + r^2/4R_o^2]^2}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right) + dt^2$$

The co-moving coordinates insure that the spatial components of velocity are zero

$$\frac{dr}{ds} = \frac{d\theta}{ds} = \frac{d\phi}{ds} = 0$$

The form of the line element now implies

$$\frac{dt}{ds} = 1$$

These conditions simplify the energy-momentum tensor

$$T^1_{\ 1} = T^2_{\ 2} = T^3_{\ 3} = -P_o \qquad T^4_{\ 4} = \rho_{00}$$

From Eq. (1), (2), and (3)

$$8\pi p_o = -\frac{1}{R_o^2} e^{-g(t)} - \ddot{g} - 3/4\,\dot{g}^2 + \Lambda \tag{4}$$

$$8\pi \rho_{oo} = \frac{3}{R_o^2} e^{-g(t)} + 3/4\,\dot{g}^2 - \Lambda \tag{5}$$

where the dots refer to time differentiation. Note that if g(t) is a constant, the equations reduce to those of the static Einstein universe.

For the density ρ_{oo} one appears justified in taking the averaged out density of energy, corresponding to galaxies, intergalactic matter and intergalactic radiation. For the pressure of the fluid it would appear reasonable to take the sum of partial pressures corresponding to the motions of galaxies, the random motions of dust or other matter in intergalactic space, and the density of intergalactic radiation.

For the galaxies the pressure corresponding to their random motions would be two-thirds of their kinetic energy per unit volume

$$P = 2/3\,\rho_k$$

from ordinary kinetic theory. For dust particles the pressure varies from two-thirds the density of the kinetic energy for nonrelativistic particles to one-third for highly relativistic particles. For radiation the pressure is generally one-third the energy density.

$$P = 1/3\, \rho_k$$

For galaxies and other slowly moving particles the kinetic energy density will be negligible compared to the energy density corresponding to the mass of the particles. Hence the total energy density is

$$3P_o + \rho_m = \rho_{oo}$$

where ρ_m corresponds to the mass of the galaxies and any intergalactic matter present. This is an approximate relation which becomes exact as the pressure due to matter can be neglected.

Combining Eqs. (4) and (5)

$$8\pi\rho_m = \frac{6}{R_o^2}\, e^{-g(t)} + 3\ddot{g} + 3\dot{g}^2 - 4\Lambda$$

C. <u>Nonconservation of energy</u>.

Define $\tau_\mu{}^\nu$ as the energy momentum density tensor

$$\tau_\mu{}^\nu = T_\mu{}^\nu \sqrt{-g}$$

Then

$$\frac{\partial \mathcal{T}_\mu{}^\nu}{\partial x^\nu} - \frac{1}{2} \mathcal{T}^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x^\mu} = 0$$

when the Christoffel symbols vanish (i.e., in flat space)

For the case $\mu = 4$

$$\frac{\partial}{\partial t}\left(\rho_{oo}\sqrt{-g}\right) + \frac{1}{2} P_o\sqrt{-g}\left(g^{11}\frac{\partial g_{11}}{\partial t} + g^{22}\frac{\partial g_{22}}{\partial t} + g^{33}\frac{\partial g_{33}}{\partial t}\right) = 0$$

since g_{44} is unity. Substituting the metric coefficients for a line element in spherical polar coordinates we get

$$\frac{\partial}{\partial t}\left[\frac{\rho_{oo} r^2 \sin\theta\, e^{3/2\, g(t)}}{(1 + r^2/4R_o^2)^3}\right] + P_o \frac{\partial}{\partial t}\left[\frac{r^2 \sin\theta\, e^{3/2\, g(t)}}{(1 + r^2/4R_o^2)^3}\right] = 0$$

Note that the proper volume measured by a local observer is given by

$$\delta V = \frac{r^2 \sin\theta\, e^{3/2\, g(t)}}{\left[1 + r^2/4R_o^2\right]^3}\, \delta r\, \delta\theta\, \delta\phi \tag{6}$$

Thus

$$\frac{d}{dt}\left(\rho_{oo}\, \delta V\right) + P_o \frac{d}{dt}\left(\delta V\right) = 0 \tag{7}$$

This equation relates the energy of any element of the fluid to the work done in adiabatic changes of volume.

It is obvious from Eq. (6) that the volume of an element of the fluid is increasing with the time if $g(t)$ is increasing with t, and decreasing when g(t) is decreasing. Also, if the pressure P_o is a positive quantity greater than zero the proper energy of every element of fluid would be decreasing when g(t) is increasing, and increasing when $g(t)$ is decreasing. Thus unless the pressure is zero, the total proper energy of the fluid will not be a constant. The principle of energy conservation can be maintained only by introducing a quantity to represent the potential energy of the gravitational field.

For convenience Eq. (7) can be written

$$\frac{d}{dt}\left(\rho_{oo}\, e^{3/2\, g(t)}\right) + P_o \frac{d}{dt}\left(e^{3/2\, g(t)}\right) = 0 \qquad (8)$$

D. Non-conservation of mass.

Recall the approximate expression previously derived for the energy density which corresponds to the mass of galaxies and intergalactic matter

$$\rho_m = \rho_{oo} - 3P_o$$

Using this equation, let us investigate the temporal dependence of matter. From Eq. (7)

$$\frac{d}{dt}\left(\rho_m \delta V\right) + 3\frac{d}{dt}\left(P_0 \delta V\right) + P_0 \frac{d}{dt}\left(\delta V\right) = 0 \tag{9}$$

The total proper mass of the galaxy is $M = \rho_m \delta V$. Then equation (9) becomes

$$-\frac{1}{M}\frac{dM}{dt} = \frac{3}{\rho_m}\frac{dP_0}{dt} + \frac{4P_0}{\rho_m \delta V}\frac{d}{dt}\delta V$$

Since δV is given by Eq. (6) this simplifies to

$$-\frac{1}{M}\frac{dM}{dt} = \frac{3}{\rho_m}\frac{dP_0}{dt} + \frac{6P_0}{\rho_m}\frac{dq}{dt} \tag{10}$$

For the special case in which the pressure is permanently equal to zero, the mass will be conserved. The mass will also be conserved for the special case where

$$3\frac{d}{dt}\left(P_0 \delta V\right) + P_0 \frac{d}{dt}\left(\delta V\right) = 0$$

This would be the case of a model containing a constant amount of matter exerting negligible pressure and where radiation exerts a pressure $P_r = \rho_r/3$.

However, one could explain any change of mass as simply the transformation of matter into energy, i.e., the mutual annihilation of electrons and protons into radiation.

E. Behavior of particles.

What is the behavior of free particles in our non-static model? Because of the principles of relativistic mechanics, the motion of free particles will be determined by the equations for a geodesic

$$\frac{d^2x^\sigma}{ds^2} + \Gamma^\sigma_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0$$

where the line element is

$$ds^2 = -\frac{e^{g(t)}}{(1 + r^2/4R_0^2)^2}\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\right) + dt^2$$

Consider first the case of a particle initially at rest with respect to the spatial coordinates r, θ, ϕ , i.e.,

$$\frac{dr}{ds} = \frac{d\theta}{ds} = \frac{d\phi}{ds} = 0 \qquad \frac{dt}{ds} = 1$$

The equations for a geodesic then become

$$\frac{d^2x^\sigma}{ds^2} + \Gamma^\sigma_{44} = 0$$

As previously shown, all values of Γ_{44}^{σ} are 0 . Hence all accelerations for this particle are 0 .

$$\frac{d^2r}{ds^2} = \frac{d^2\theta}{ds^2} = \frac{d^2\phi}{ds^2} = \frac{d^2t}{ds^2} = 0$$

This particle will thus remain stationary for all time. This result is hardly unexpected considering our choice of co-moving coordinates.

Introduce now the more general case of particles having some arbitrary initial velocity. It will be convenient to consider first the geodesic equation with $\sigma = 4$.

$$\frac{d^2t}{ds^2} + \Gamma_{\mu\nu}^{4} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

Substituting the values of $\Gamma_{\mu\nu}^{4}$ for our particular line element one obtains

$$\frac{d^2t}{ds^2} + \frac{1}{2} e^{\mu} \dot{\mu} \left(\frac{dr}{ds}\right)^2 + \frac{1}{2} e^{\mu} \dot{\mu} r^2 \left(\frac{d\theta}{ds}\right)^2 + \frac{1}{2} e^{\mu} \dot{\mu} r^2 \sin^2\theta \left(\frac{d\phi}{ds}\right)^2 = 0$$

where

$$e^{\mu} = \frac{e^{g(t)}}{(1 + r^2/4R_o^2)^2}$$

This can be rewritten as

$$\frac{d^2t}{ds^2} + \frac{1}{2}\frac{dg}{dt}\left[\left(\frac{dt}{ds}\right)^2 - 1\right] = 0$$

or

$$\frac{2\,\frac{dt}{ds}\,\frac{d}{dt}\left(\frac{dt}{ds}\right)}{\left(\frac{dt}{ds}\right)^2 - 1} = -\frac{dg}{dt}$$

This can be easily integrated to give

$$\left(\frac{dt}{ds}\right)^2 - 1 = Ae^{-g(t)}$$

where A is a constant of integration.

The line element can be written in the following form

$$\left(\frac{ds}{dt}\right)^2 = 1 - \frac{e^{g(t)}}{(1 + r^2/4R_0^2)^2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right]$$
$$= 1 - u^2/c^2$$

where u is the velocity of the particle as seen by an observer at rest with respect to r, θ, ϕ and who uses his own determinations of increments in proper time and proper distance.

$$dt_0 = dt \qquad dl_0 = \frac{e^{\frac{1}{2}g(t)}}{(1 + r^2/4R_0^2)}\,dr \qquad \text{etc.}$$

Substituting into the above equations

$$\frac{u^2/c^2}{1-u^2/c^2} = Ae^{-g(t)}$$

Thus, if g(t) is increasing in time, the velocities of free particles will be decreasing and vice versa.

By solving these equations the energy of free particles can be discussed. One gets

$$E = \frac{E_0}{\sqrt{1-u^2/c^2}} = E_0\sqrt{1+Ae^{-g(t)}}$$

Thus the energy of free particles will decrease as g(t) is increasing.

F. <u>Behavior of Light Rays</u>.

The equations for a geodesic would be applicable to the motion of light rays in the case $ds = 0$. For the special case in which the ray of light moves only in the radial direction we have $\frac{d\theta}{dt} = \frac{d\Phi}{dt} = 0$ and

$$\frac{dr}{dt} = \pm e^{-\frac{1}{2}g(t)}\left[1 + \frac{r^2}{4R_0^2}\right]$$

Using methods identical to those of the preceding section one can show that if a light ray moves initially in the radial direction it will permanently continue its motion in a radial direction.

Integrate over the time interval needed for a ray to travel between the origin and some point r

$$\int_0^r \frac{dr}{1 + r^2/4R_0^2} = \int_{t_1}^{t_2} e^{-\frac{1}{2}g(t)}\, dt$$

$$2R_0 \tan^{-1} \frac{r}{2R_0} = \int_{t_1}^{t_2} e^{-\frac{1}{2}g(t)}\, dt \qquad (11)$$

To evaluate the right hand integral let us assume g(t) is linear in t

$$g = 2kt$$

The integral becomes

$$r = 2R_0 \tan \frac{e^{-kt_1} - e^{-kt_2}}{2kR_0}$$

In the case of a closed, always expanding model of the universe this relation leads to an interesting restriction. Assuming the linear dependence of g(t) and a real radius $R = R_0 e^{kt}$ one sees that light can always be received at the origin at any finite time t_2 if a sufficiently early starting time t_1 is chosen. However, light which leaves the origin at t_1 will reach a maximum coordinate distance

$$r = 2R_0 \tan \frac{e^{-kt_1}}{2kR_0}$$

even at $t_2 = \infty$. Depending on k and R_0 there exists a specific starting time after which light cannot travel completely around the model. An observer could then, in principle, obtain information about sufficiently early states of the model, but even by waiting an infinite amount of time he could not receive information on their subsequent behavior.

G. Doppler Effect.

Let us consider the observer as fixed permanently at the origin of our coordinate system. Differentiating equation (11) one gets

$$e^{-\frac{1}{2}g_2}\,\delta t_2 - e^{-\frac{1}{2}g_1}\,\delta t_1 = \frac{1}{1 + r^2/4R_0^2}\left(\frac{dr}{dt}\right)\delta t_1$$

connecting the "time" interval δt_1 between the departure of two wave crests from the source to the interval δt_2 between their arrival at the origin, where g_1 and g_2 are the values of

$g(t)$ at $t = t_1$ and $t = t_2$ respectively. Since $\frac{dr}{dt}$ is the radial "coordinate velocity" of the source at the time of emission we can write

$$\frac{1}{1 + r^2/4R_o^2}\left(\frac{dr}{dt}\right) = e^{-\frac{1}{2}g_1}\frac{u_r}{c}$$

where u_r is the radial component of the velocity of the source as measured by our observer at rest. This gives

$$e^{-\frac{1}{2}g_2}\delta t_2 = e^{-\frac{1}{2}g_1}\delta t_1 + e^{-\frac{1}{2}g_1}\frac{u_r}{c}\delta t_1$$

The proper time interval δt_1^o between the emission of these wave crests as measured by an observer moving with the source is related to δt_1 by

$$\delta t_1^o = \left\{-\frac{e^{g_1}}{(1 + r^2/4R_o^2)^2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right] + 1\right\}\delta t_1$$

This can be rewirtten as

$$\delta t_1^o = \left(1 - u^2/c^2\right)^{1/2}\delta t_1$$

For the observer at rest the proper time between the reception of the wave crests is

$$\delta t_2^\circ = \delta t_2$$

Equating the ratio of periods of the emitted and received light to the ratio of the corresponding wave lengths one gets

$$\frac{\lambda + \delta\lambda}{\lambda} = \frac{\delta t_2^\circ}{\delta t_1^\circ} = \frac{e^{\frac{1}{2}(g_2 - g_1)}}{(1 - u^2/c^2)^{1/2}}\left(1 + \frac{u_r}{c}\right)$$

Consider only the term connected to the general expansion of the model

$$\frac{\lambda + \delta\lambda}{\lambda} = e^{\frac{1}{2}(g_2 - g_1)}$$

introducing the radius

$$R = R_\circ e^{\frac{1}{2}g}$$

then

$$\frac{\delta\lambda}{\lambda} = \frac{R_2 - R_1}{R_1}$$

where R_1 is the radius of the model at time of emission and R_2 is the radius when the light arrives at the origin. The red shift is thus closely correlated with the general expansion of the model.

This dependence can be clarified by introducing the total proper distance from observer to source.

$$\ell_1 = e^{\frac{1}{2}g_1}\int_0^r \frac{dr}{1 + r^2/4R_0^2} \qquad \ell_2 = e^{\frac{1}{2}g_2}\int_0^r \frac{dr}{1 + r^2/4R_0^2}$$

as determined at times t_1 and t_2. Then

$$\frac{\lambda + \delta\lambda}{\lambda} = \frac{\ell_2}{\ell_1} = 1 + \frac{\ell_2 - \ell_1}{\ell_1}$$

where $\ell_2 - \ell_1$ is the increase in proper distance from the source to the observer that occurs during the time it takes light to travel from one to the other. In a first approximation the time of travel equals the proper distance

$$\frac{\lambda + \delta\lambda}{\lambda} \simeq 1 + \frac{\delta\ell}{\delta t} \simeq 1 + \frac{u}{c}$$

where u is approximately the velocity of recession. Because of the homogeneity of the model, observers at rest would see similar red shifts in other portions of the world; there can be nothing unique about our initial coordinate.

H. Change of Doppler Effect with distance.

How does the Doppler Effect change as we go to more distant galaxies? Differentiate the previous expression $\frac{\delta\lambda}{\lambda} = e^{\frac{1}{2}(g_2 - g_1)} - 1$ with respect to r, the distance to the source. Note that g_2 is a constant since this is the value of g(t) at the origin when the light is received. On the other hand g_1 is a variable since we are interested in going to greater distances r where the light must be emitted at an earlier time t_1 in order to reach the origin at t_2. Thus

$$\frac{d}{dr}\left(\frac{\delta\lambda}{\lambda}\right) = -\frac{1}{2}e^{\frac{1}{2}(g_2-g_1)}\frac{dg_1}{dt}\frac{dt}{dr}$$

where dt is the change in the time of emission which corresponds to a change dr. From our expression for the line element

$$\frac{dt}{dr} = \frac{e^{\frac{1}{2}g_1}}{1+r^2/4R_o^2}$$

and

$$\frac{d}{dr}\left(\frac{\delta\lambda}{\lambda}\right) = \frac{e^{\frac{1}{2}g_2}}{1+r^2/4R_o^2}\frac{\dot{g}_1}{2}$$

As long as $r \ll R_o$ and the change in $\dot{g}_1$ is small the derivative is approximately a constant. Hence for "reasonably small" r we have

$$\frac{\delta\lambda}{\lambda} \propto R \qquad \text{(Hubble's constant)}$$

At large distances the relationship is no longer linear. This non-linearity offers some possibility of distinguishing between various models.

By choosing the following line element

$$ds^2 = - e^{g(t)}\left[\frac{d\bar{r}^2}{1-\bar{r}^2/R_o^2} + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta \, d\phi^2\right] + dt^2$$

where $\bar{r} = \dfrac{r}{1 + r^2/4R_o^2}$ the derivative becomes

$$\frac{d}{d\bar{r}}\left(\frac{\delta\lambda}{\lambda}\right) = \frac{e^{\frac{1}{2}g_2}}{\left(1-\bar{r}^2/R_o^2\right)^{1/2}} \frac{\dot{g}_1}{2}$$

I. Closed Models.

Consider a closed model with a real radius R_o and assume the density ρ_{oo} and pressure P_o can only be zero or positive. From equations (5) and (8) one can write

$$\frac{d}{dt}\left(\rho_{oo}\, e^{\frac{3}{2}g(t)}\right) + P_o \frac{d}{dt}\left(e^{\frac{3}{2}g(t)}\right) = 0 \qquad (12)$$

$$8\pi\rho_{oo} = \frac{3}{R_o^2} e^{-g(t)} + \frac{3}{4}\left(\frac{dg}{dt}\right)^2 - \Lambda \qquad (13)$$

and one recalls $R = R_o\, e^{\frac{1}{2}g(t)}$.

The first equation then becomes

$$\frac{d}{dt}\left(\rho_{oo} R^3\right) + P_o \frac{d}{dt}\left(R^3\right) = 0$$

which becomes

$$\frac{d}{dR}\left(\rho_{oo} R^3\right) = -3 P_o R^2 \tag{14}$$

and

$$\frac{d\rho_{oo}}{dR} = -\frac{3\left(\rho_{oo} + P_o\right)}{R} \tag{15}$$

Thus $\rho_{oo} R^3$ and ρ_{oo} can only decrease or remain constant as R increases. The density of the fluid will go to zero if the radius goes to infinity so all ever-expanding models will finally have the properties of the de Sitter model: $\rho_{oo} = P_o = 0$

Substitute the exponential dependence of $R(R = R_o\, e^{\frac{1}{2}g(t)}$ into Eq. (13).

$$8\pi\rho_{oo} = \frac{3}{R_o^2}\left(\frac{R_o}{R}\right)^2 + \frac{3}{4}\,\frac{4}{R^2}\left(\frac{dR}{dt}\right)^2 - \Lambda$$

then

$$\frac{dR}{dt} = \pm\left[\frac{8\pi\rho_{oo} R^2}{3} + \frac{\Lambda R^2}{3} - 1\right]^{1/2}$$

Since the rate of change of the radius of our model must be real, the quantity inside the bracket must be positive or zero. If we consider any given value of the cosmological constant Λ this leads to a restriction on the radius:

$$\frac{3}{R^2} - 8\pi\rho_{00} \geq \Lambda$$

Let us define a quantity $Q(R)$

$$Q(R) \equiv \frac{3}{R^2} - 8\pi\rho_{00} = \frac{1}{R^2}\left(3 - \frac{8\pi\rho_{00}R^3}{R}\right) \qquad (16)$$

and investigate its behavior. The extrema of $Q(R)$ are found by differentiating

$$\frac{dQ}{dR} = -\frac{6}{R^3} + \frac{8\pi d\rho_{00}}{dR} = 0$$

Using equation (14) this becomes

$$\frac{dQ}{dR} = -\frac{6}{R^3} + \frac{24\pi(\rho_{00} + P_0)}{R} = 0 \qquad (17)$$

Substituting the definition of $Q(R)$ into this expression one finds

$$Q(R) = \frac{1}{R^2} + 8\pi P_0 \quad > \quad 0 \tag{18}$$

This is an equation for the value of $Q(R)$ at an extrema or point of inflection. The second derivative becomes

$$\frac{d^2Q}{dR^2} = \frac{18}{R^4} - \frac{24\pi(\rho_{oo} + P_0)}{R^2} + 24\pi \frac{d\rho_{oo}}{dR} + 24\pi \frac{dP_0}{dR}$$

Using equations (16) and (17)

$$\frac{d^2Q}{dR^2} = \frac{18}{R^4} - \frac{6}{R^4} - 24\pi \frac{3(\rho_{oo} + P_0)}{R} + 24\pi \frac{dP_0}{dR}$$

$$= \left[\frac{-(\rho_{oo} + P_0)}{R} + \frac{dP_0}{dR} \right] 24\pi$$

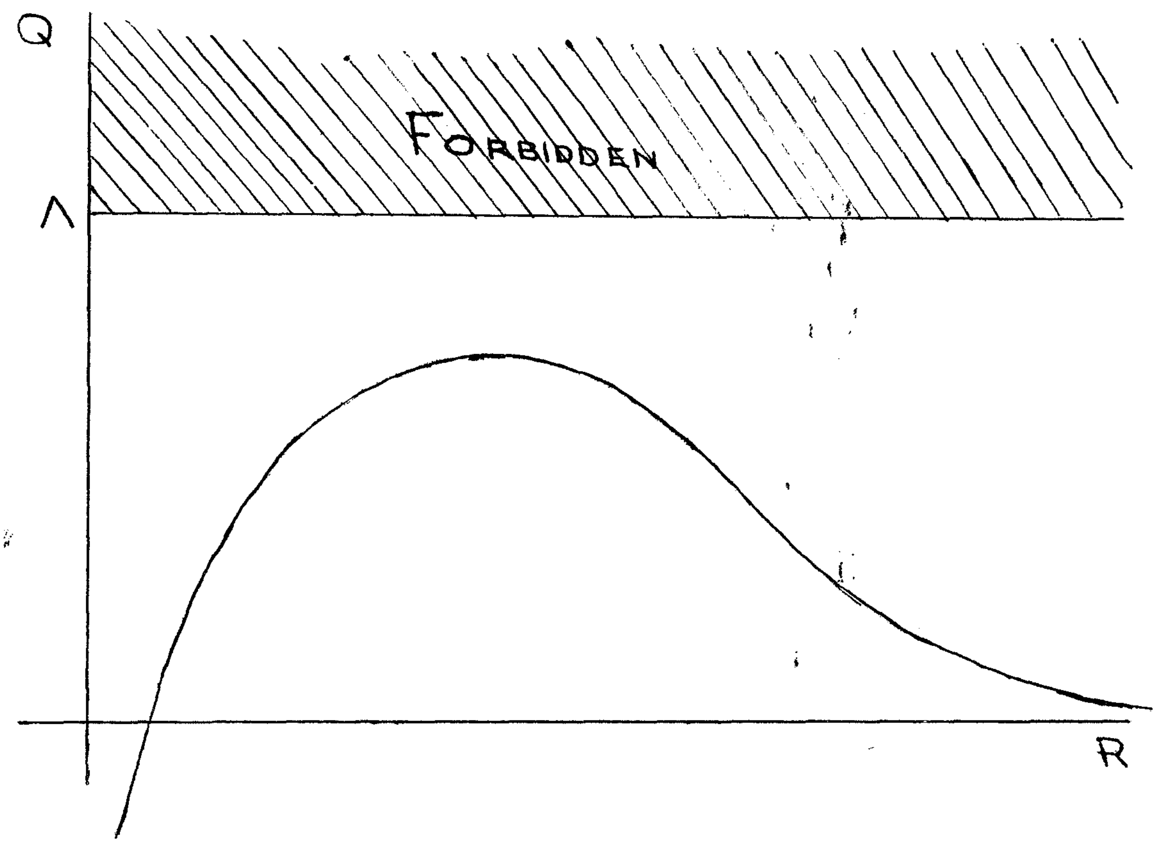
Hence

$\frac{dP_0}{dR} < \frac{\rho_{oo} + P_0}{R} \implies$ a maximum

$\frac{dP_0}{dR} = \frac{\rho_{oo} + P_0}{R} \implies$ a point of inflection

$\frac{dP_0}{dR} > \frac{\rho_{oo} + P_0}{R} \implies$ a minima

Unless we assume the pressure of the fluid can be increasing as the model expands, there can be no minima or points of inflection. We must have a curve with one maximum and no minima.



The general features of the curve can be easily explained. From Eq. (14) it is apparent that $\frac{d}{dR}\left(\rho_{00} R^3\right)$ can only decrease or remain constant as R increases. From Eq. (17) one sees that Q rises asymptotically from minus infinity at R = 0 if we exclude the case of a completely empty (ρ_{00} = 0) model.

Q will continue to increase as R increases at least until $Q > 0$ since Eq. (18) requires Q be positive at an extrema or point of inflection. If R continues to increase, Q must exhibit a maxima. Finally as R becomes very large, Q must approach zero asymptotically.

Let us now discuss what models of the universe are possible subject to the preceding restrictions on Q.

a) Monotonic universe of type M_1 for $\Lambda > \Lambda_E$.

Denote the maximum value of Q by Λ_E. For the Λ = const. line makes no intersections with the critical curve so the model is that of an ever-expanding type which at some singular state $R_S \geq 0$ and proceeds to the final state of an empty de Sitter universe as $R \to \infty$. This is a monotonic universe of the first type, M_1. As a model for the physical universe it has the disadvantage of spending an infinitesimal portion of its existence in a condition which differs from a completely empty de Sitter universe. Since our observations of the universe presumably give us a good idea of the conditions that would be found anywhere at any time, this model can be ruled out.

b) Asymptotic universes with $\Lambda = \Lambda_E$.

From Eq. (18) we have

$$\Lambda_E = 8\pi P_E + \frac{1}{R_E^2}$$

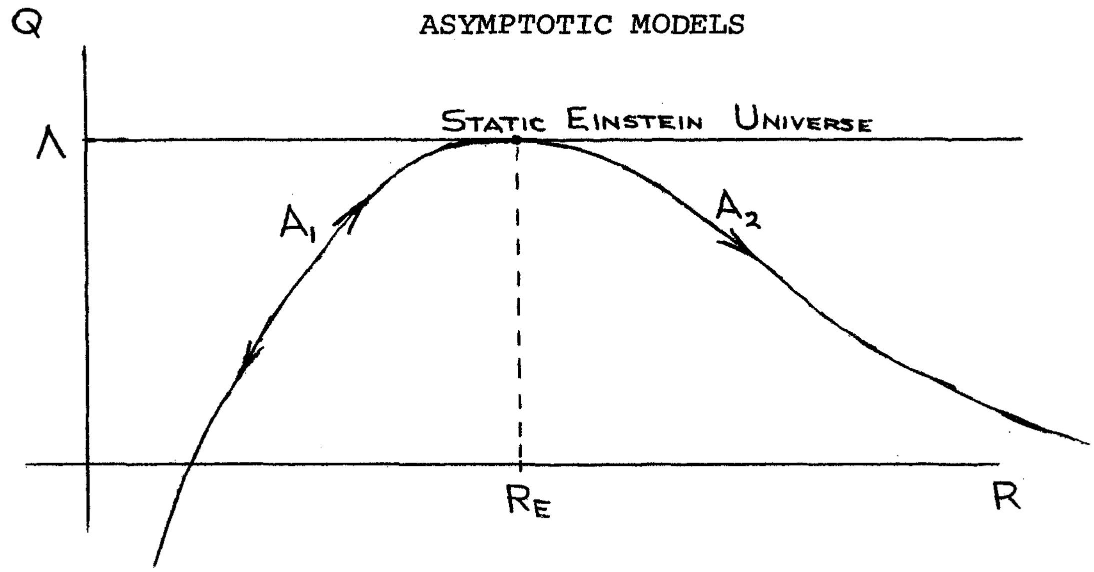
at the maximum point of Q, where P_E and R_E are the pressure and the radius at that point. If we consider a static universe $\left(\frac{dR}{dt} = 0\right)$ at this radius R_E and cosmological constant Λ_E

$$8\pi \rho_E = \frac{3}{R_E^2} - \Lambda_E$$

These are the equations for pressure and density of a static Einstein universe. Therefore a stat c Einstein universe could exist with a radius corresponding to the maximum value of Q.

With $\Lambda = \Lambda_E$ two types of behavior are possible. The first type begins from a singular state $R_s < R_E$ and asymptotically approaches the static Einstein universe at $R = R_E$ where both $\frac{dR}{dt}$ and $\frac{d^2R}{dt^2}$ would become zero. At times earlier than the singular state this model would contract from larger radii down to $R = R_S$. This model is an asymptotic universe of the first type, A_1.

The remaining type of behavior for $\Lambda = \Lambda_E$ is given by a model which can be regarded as having asymptotically started from a static Einstein universe with $R = R_E$ at an infinite time in the past and expanded monotonically into an empty de Sitter universe. This is an asymptotic model of the second type, A_2.



As models of the physical universe both of these types have the same disadvantage as type M_1; they spend only a small fraction of their existence in conditions which approximate our observations of the universe. Type A_2 has the interesting feature of originating from a non-singular state of finite volume at an infinite time in the past.

c) Monotonic Universes of type M_2 and oscillating universes of types O_1 and O_2 for $0 < \Lambda < \Lambda_E$

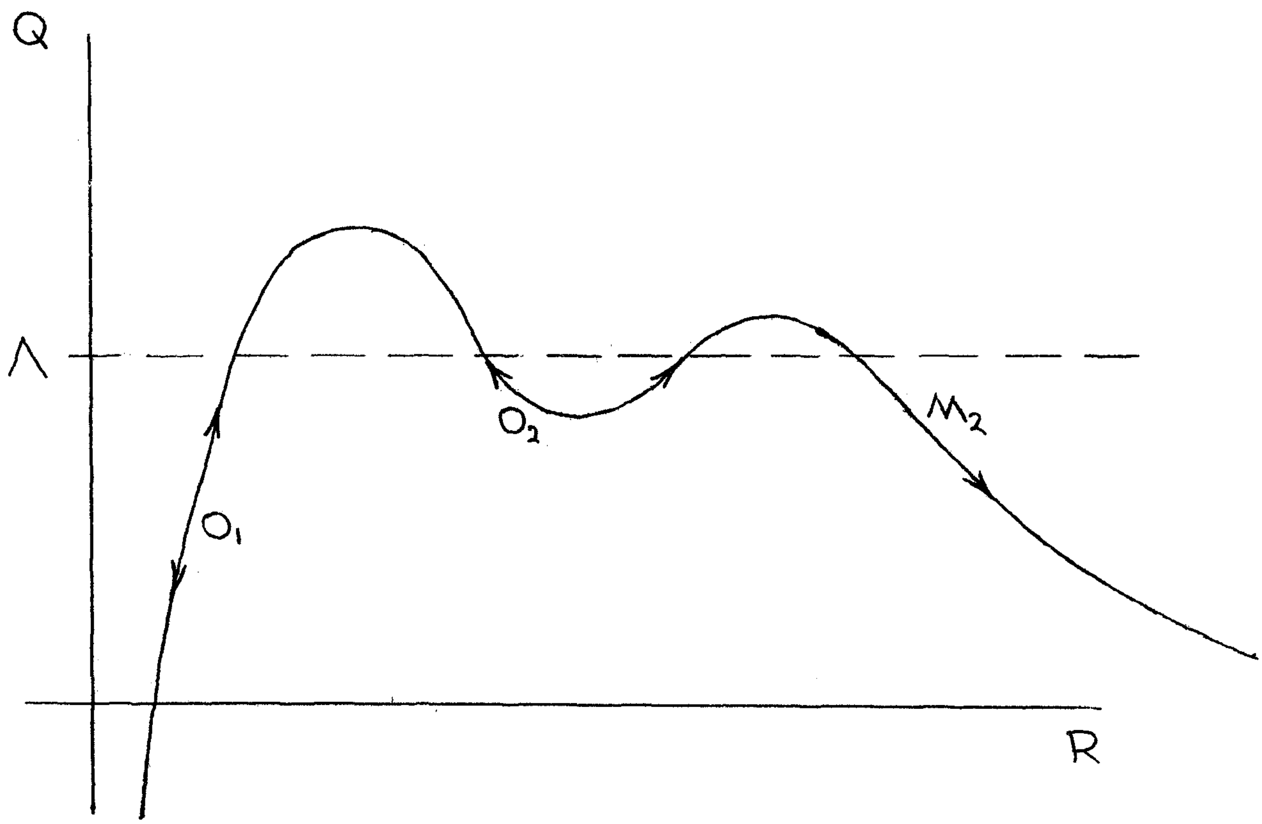
For Λ between 0 and Λ_E two types of behavior are possible. One type of behavior concerns those models that expand continuously into the future from a point on the critical curve at $R_1 > R_E$. This model begins at a finite radius and expands monotonically to an empty de Sitter universe. It is designated as a monotonic universe of the second type, M_2. The main disadvantage of this model is the same as model M_1; it spends all but an infinitesimal portion of its existence in a state unlike that which we observe.

The second type of behavior is characterized by models which expand from a singular state at $R_S < R_E$ to a maximum radius given by the intersection of the Λ = const. curve with the critical curve. Once the model reaches the maximum radius it will begin to contract back to the singular state from which expansion will begin again. This is an oscillating universe of the first kind, O_1. It has the advantage of spending its entire life in a condition where there is a finite density of matter. However, it has a disadvantage since the singular state at the lower limit of contraction is not described by the present equations.

If there exist conditions in the universe which permit the pressure to increase as the universe expands the critical curve can have a minima. Such a minima gives rise to an interesting type of behavior. If Λ assumes a value between this minima and a secondary maxima of Q, there can be an oscillation between a true minimum and a maximum radius. For reversible behavior this gives rise to a strictly periodic behavior without singular states. This model is an oscillating universe of the second kind, O_2 .

This model appears to have substantial advantages over previous examples. However, the assumption that pressure can increase as the universe expands seriously limits the usefulness of this model.

OSCILLATING MODELS



d) Oscillating universe of type O_1 for $\Lambda \leqslant 0$.

Finally the case for $\Lambda \leqslant 0$ must be discussed. It is obvious that the only behavior possible would be an oscillation of the type O_1 between a singular state at the lower limit and the maximum radius. This would have the same advantages and disadvantages previously mentioned.

This model represents a very important case, that of $\Lambda = 0$. At the present time there exists no evidence that $\Lambda \neq 0$. If, indeed, $\Lambda = 0$ a closed universe could only be type O_1 .

The cosmological constant was originally postulated to obtain a universe with a finite density of matter in the static situation. In view of the non-static models this assumption is no longer necessary. Such a cosmological constant must be "reasonably small" or its effect would be noticeable in planetary motions.

J. Open Models.

It is also possible that the universe is infinite in extent with R_o either imaginary or infinite. In this case the possible types of behavior are quite restricted.

Consider Eqs. (12) and (13)

$$\frac{d}{dt}\left(\rho_{oo}\, e^{3/2\, g(t)}\right) + P_o \frac{d}{dt}\left(e^{3/2\, g(t)}\right) = 0 \qquad (12)$$

$$8\pi\rho_{oo} = \frac{3}{R_o^2} e^{-g(t)} + \frac{3}{4}\left(\frac{dg}{dt}\right) - \Lambda \tag{13}$$

For an open model of the universe the radius, either imaginary or infinite, is a quantity of limited usefulness, so there is no advantage of introducing it. Re-express Eq.(12) as:

$$\frac{d}{dg}\left(\rho_{oo} e^{3/2 g}\right) = -\frac{3}{2} P_o e^{3/2 g}$$

and

$$\frac{d\rho_{oo}}{dg} = -\frac{3}{2}(\rho_{oo} + P_o)$$

whcih show that $\rho_{oo} e^{3/2 g}$ and ρ_{oo} can only decrease or remain constant as g increases, if we again assume that the pressure in the model doesn't increase as the model expands.

Eq. (13) can be written:

$$\left[\frac{d}{dt}\left(e^{1/2 g(t)}\right)\right]^2 = \frac{8\pi\rho_{oo}}{3} e^{g} + \frac{\Lambda}{3} e^{g} - \frac{1}{R_o^2}$$

By our assumptions R_o is either imaginary or infinite so this equation can be re-expressed:

$$\frac{d}{dt}\, e^{\frac{1}{2}g(t)} = \pm \left[\frac{8\pi\rho_\infty}{3} e^{g} + \frac{\Lambda}{3} e^{g} + A^2 \right]^{1/2}$$

where A is real quantity that is zero if R_0 is infinite. The quantity in the bracket must be positive so

$$-3A^2 e^{-g} - 8\pi\rho_\infty \leqslant \Lambda$$

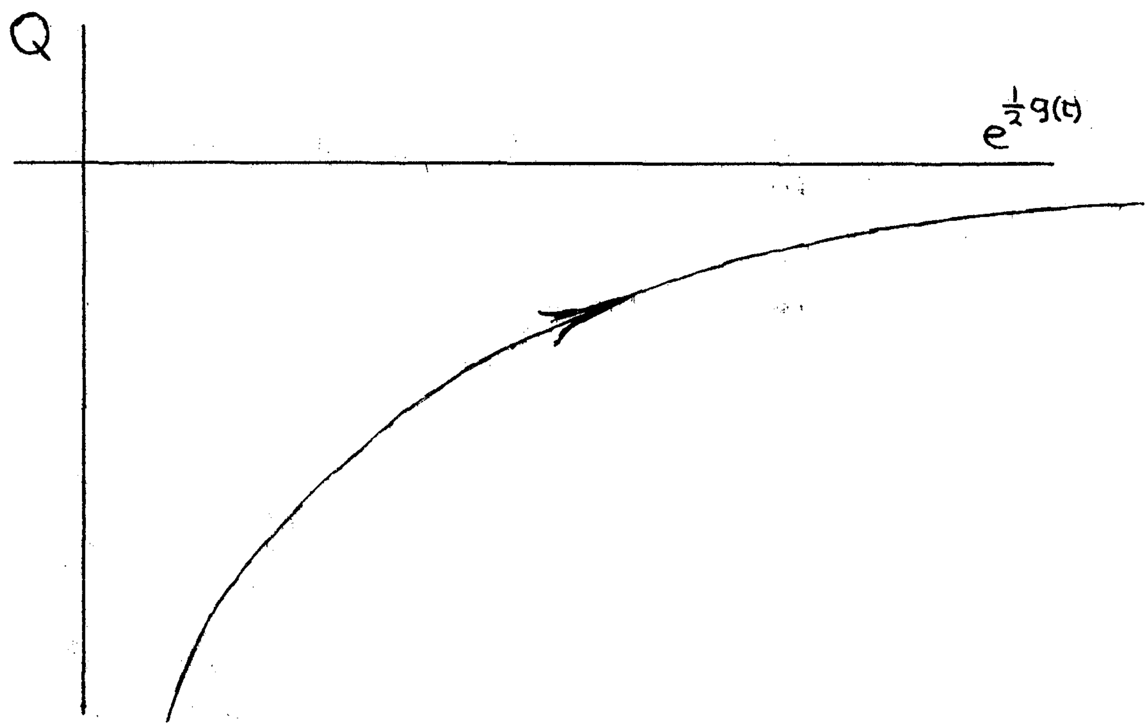
is a necessary restriction on g if the behavior of the model is to be real. Also

$$-3A^2 e^{-g} - 8\pi\rho_\infty = 0$$

is the condition for a reversal in the direction of the rate of change of g with t. Define Q as

$$Q \equiv -3A^2 e^{-g} - 8\pi\rho_\infty$$

Q is always negative, asymptotically approaching $Q = -\infty$ as $e^{\frac{1}{2}g}$ goes to zero and $Q = 0$ as $e^{\frac{1}{2}g}$ goes to infinity, without any maxima, minima, or points of inflection.



Clearly, two types of behavior are possible. The first type occurs for $\Lambda \geq 0$ and would consist of the monotonic increase of $e^{\frac{1}{2}g}$ from a singular state to infinity. Essentially this is an M_1 type universe which ends in an empty de Sitter universe, including the possibility of a Euclidean Space with $\Lambda = 0$.

The second type of behavior occurs for $\Lambda < 0$. In this case $e^{\frac{1}{2}g}$ would proceed from a singular state to a maximum and return, giving rise to an oscillating universe of type O_1.

NEWTONIAN COSMOLOGY

The first attempts to understand cosmology on a rigorous basis were made during the nineteenth century with the Newtonian theory. It is ironic that these attempts failed not because of any fault of Newtonian theory but because of the assumption of a static universe. Following this failure interest in cosmology decreased. Not until Einstein began exploring the cosmological consequences of general relativity in 1916 did interest awaken. Then followed a 15 year period in which Einstein's relativistic cosmology was consolidated and extended. Not until 1934 did Milne and McCrea attack the Newtonian problem. They showed that in many ways Newtonian cosmology was very similar to relativistic cosmology. The Newtonian formulation of cosmology is very useful because it reveals much of the essential features of relativistic cosmology without the mathematical difficulties.

Newtonian theory is fully accepted as is the cosmological principle. Thus our "universe" is both isotropic and homogeneous. There is a uniform even-flowing Newtonian time and so the relativistic problem of clock synchronization does not arise.

Consider an observer 0. He observes the motion of a particle relative to him as a function of $\vec{r}$ and t : $v(\vec{r},t)$.

The density and pressure are: $\rho(\vec{r},t)$, $P(\vec{r},t)$

A second observer 0' will see the particle move with velocity $\vec{V}'(\vec{r},t)$. He sees a density $\rho(\vec{r}',t)$ and pressure $P(\vec{r}',t)$. Since density and pressure are defined independently of 0 observer there is no need for a prime on these quantities. The cosmological principle now demands that $\vec{V}', \rho$ and P should be the <u>same</u> functions of $\vec{r}'$ and t as $\vec{V}, \rho$, P are of $\vec{r}, t$. Otherwise the two observers would have different pictures of what is going on.

Assume now $t = 0$ for simplicity and that the vector 00' is $\vec{a}$. Then $\vec{r}' = \vec{r} - \vec{a}$ and

$$\vec{V}'(\vec{r}-\vec{a}) = \vec{V}(\vec{r}) - \vec{V}(\vec{a}) \qquad \rho(\vec{r}-\vec{a}) = \rho(\vec{r}) \qquad P(\vec{r}-\vec{a}) = P(\vec{r})$$

By the cosmological principle $\vec{V}'$ is the same function of its argument as $\vec{V}$. Thus

$$\vec{V}(\vec{r}-\vec{a}) = \vec{V}(\vec{r}) - \vec{V}(\vec{a})$$

From this we see that $\vec{V}$ is a linear vector function of its argument so

$$\vec{V} = A\,\vec{r} \qquad \text{where } A \text{ is independent of } r\,.$$

The assumptions of isotropy and homogeneity insure that the density and pressure are independent of position. Hence

$$\vec{V} = f(t)\vec{r}$$
$$\rho = \rho(t)$$
$$P = P(t)$$

The velocity can be integrated to give

$$\vec{r} = R(t)\,\vec{r}_o$$

where R(t) satisfies

$$\frac{1}{R}\,\frac{dR}{dt} = f(t) \qquad R(t_o) = 1$$

and $\vec{r}_o$ is the position vector of the particle at time t_o. From this we see that the only motions compatible with homogeneity and isotropy are uniform expansion and contraction with a time dependent scale factor.

Combining the equation of continuity with the previous equations

$$0 = \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\vec{v}) = \frac{\partial \rho}{\partial t} + 3\,\rho(t)\,f(t)$$

Integrating this and using the definition of R(t):

$$\rho(t) = \frac{\rho(t_o)}{R^3(t)} \quad .$$

This equation is the obvious condition that if the universe is scaled up by a factor of R all volumes are increased by R^3 and the density is correspondingly lowered.

Let us consider our space as a fluid. Euler's equations of hydrodynamics can be applied:

$$\frac{d\vec{v}}{dt} + \frac{1}{\rho}(\text{grad } P) - \vec{F} = \vec{r}\left(\frac{df}{dt} + f^2\right) - \vec{F}$$

where $\vec{F}$ is the body force per unit mass (gravitation).

The evaluation of the gravitational force in an infinite system is clearly rather ambiguous. In this model we shall use Poisson's Eqn.

$$\operatorname{div} \vec{F} = -4\pi\gamma\rho$$

Take the divergence of Euler's equations -

$$3\left(\frac{df}{dt} + f^2\right) = -4\pi\gamma\rho \qquad (1)$$

This result could also have been derived by assuming that the effective gravitational force on a particle viewed from 0 is due entirely to the sphere of matter with center at 0 and its surface passing through the particle.

$$F = -\frac{4\pi}{3}\gamma\rho\vec{r}$$

Using the definition of $R(t)$, the equation for $\rho(t)$ and substituting in equation (1) we get:

$$R^2\frac{d^2R}{dt^2} + \frac{4\pi}{3}\gamma\rho(t_o) = 0 \qquad (2)$$

From this equation it is obvious that a static universe ($R=1$) is impossible except when the density vanishes. It is here that nineteenth century cosmology floundered. Various proposals attempted to overcome this difficulty by postulating ad hoc changes in the law of gravitation. Due to this arbitrariness these changes found little favor.

In general relativity an equation exactly analogous to (2) occurs where the alteration of the law of gravity is not arbitrary. The Newtonian analogue of the relativistic procedure is well defined and consists of introducing into the definition of $\vec{F}$ a term proportional to the distance and independent of the density. Thus F becomes

$$F = -\frac{4\pi}{3}\gamma\rho(t)\vec{r} + \frac{1}{3}\Lambda\vec{r}$$

where Λ, the cosmological constant, has dimensions $(\text{time})^{-2}$ and the factor 1/3 is introduced for later convenience.

Equation (2) now becomes

$$R^2 \frac{d^2R}{dt^2} + \frac{4\pi}{3}\gamma\,\rho(t_o) - \frac{1}{3}\Lambda R^3 = 0 \tag{3}$$

The cosmological constant was first introduced to obtain a static model of the universe. However, the solutions of Eq. (3) for various values of Λ have attracted much interest. The integrated form of Eq. (3)

$$\left(\frac{dR}{dt}\right)^2 = \frac{C}{R} - k + \frac{1}{3}\Lambda R^2 = G(R) \tag{4}$$

is identical in form to the relativistic equation. k is a constant of integration which has dimensions $(\text{time})^{-2}$ and $3C = 8\pi\gamma\rho(t_o)$.

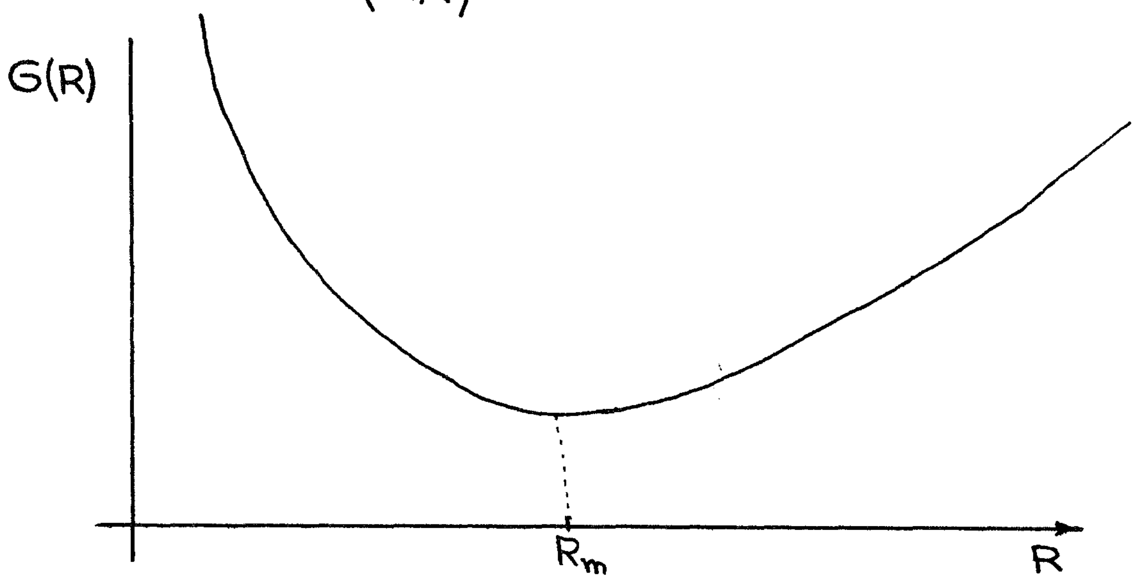
Equation (4) can be integrated in terms of elliptic functions but it is more illustrative to look at specific cases. The direction of time will be chosen to lead to an expanding universe. The time origin is arbitrary and will be chosen for convenience. The parameters Λ and k are also arbitrary but C must be positive since we are dealing only with positive mass densities.

Case I: $k < 0$

i) $\Lambda > 0$

Here $G(R)$ is a positive function of R since $\frac{dR}{dt}$ must be real and $\left(\frac{dR}{dt}\right)^2$ positive. The minimum occurs at

$$R_m = \left(\frac{3C}{2\Lambda}\right)^{1/3}$$



For small t we assume that R is also small. Hence terms in k and $\frac{1}{3}\Lambda R^2$ can be neglected compared to C/R.

$$\frac{dR}{dt} = \frac{C^{1/2}}{R^{1/2}} \Rightarrow R^{1/2}\, dR = C^{1/2}\, dt$$

$$R \sim \left(\frac{9Ct^2}{4}\right)^{1/3}$$

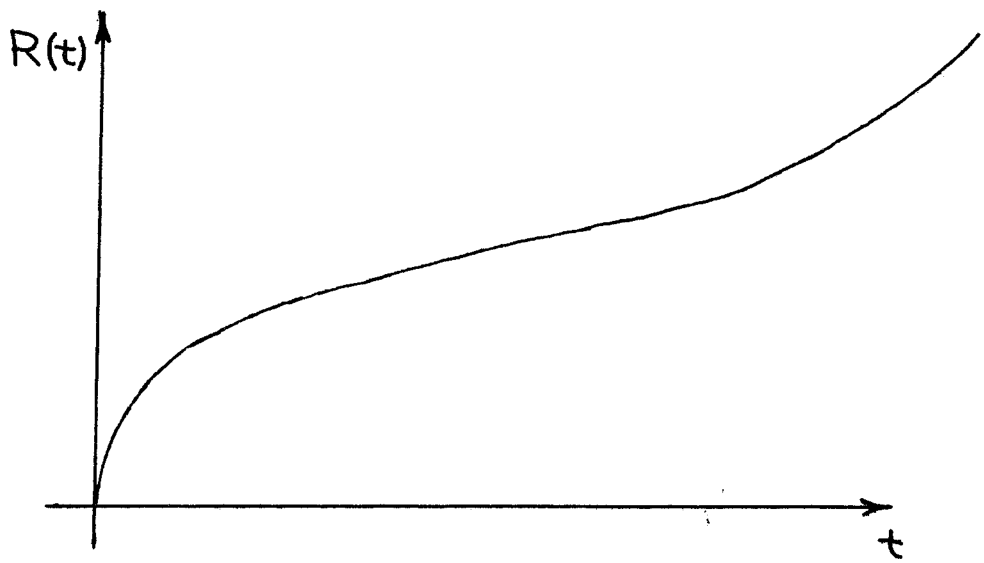
As $R \to R_m$ the rate of expansion slows down and approaches a constant value

$$\left(\frac{dR}{dt}\right)^2 = \frac{C}{R_m} - k + \frac{1}{3}\Lambda R_m^{\ 2} \Rightarrow R_m\left(\frac{dR}{dt}\right)^2 = C - k + \frac{1}{3}\Lambda\frac{3C}{2\Lambda} = \frac{3}{2}C - k$$

$$\left(\frac{dR}{dt}\right) \sim \left[\frac{\left(\frac{3}{2}C-k\right)}{\left(3C/2\Lambda\right)^{1/3}}\right]^{1/2}$$

For large t R is large so terms C/R and -k can be neglected.

$$\frac{dR}{dt} = \left(\frac{1}{3}\Lambda\right)^{1/2} R \Rightarrow \quad R \sim \exp\left[t\left(\frac{1}{3}\Lambda\right)^{1/2}\right]$$



ii) $\Lambda = 0$

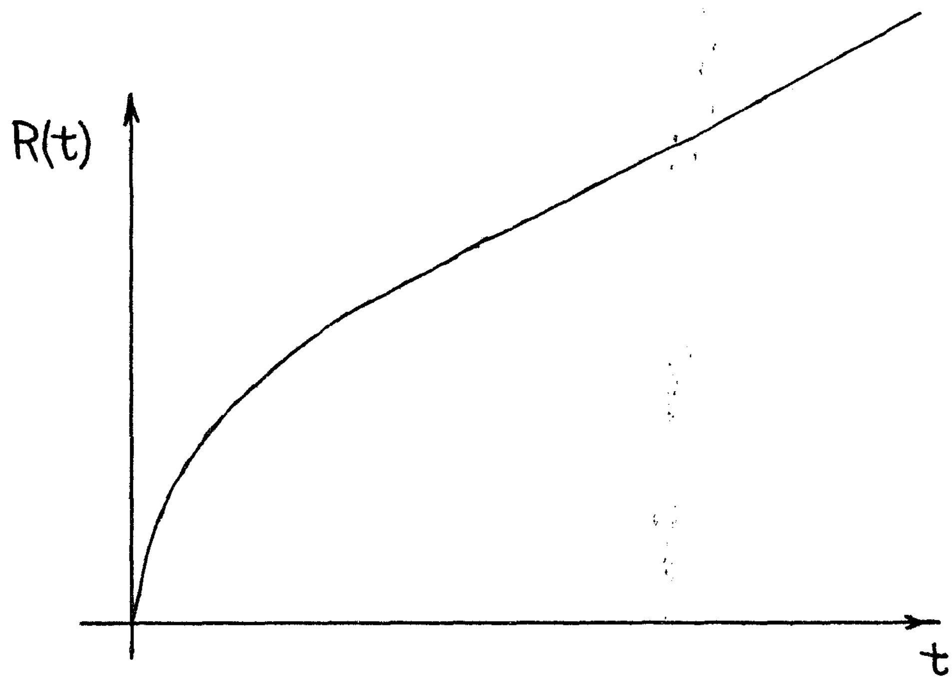
G(r) is a positive decreasing function of R.

For small t we again consider only the term C/R . Hence

$$R \sim \left(\frac{9Ct^2}{4}\right)^{1/3}$$

The rate of expansion slows down continuously. For large t

$$\frac{dR}{dt} = (-k)^{1/2} \Rightarrow R \sim (-k)^{1/2} t$$

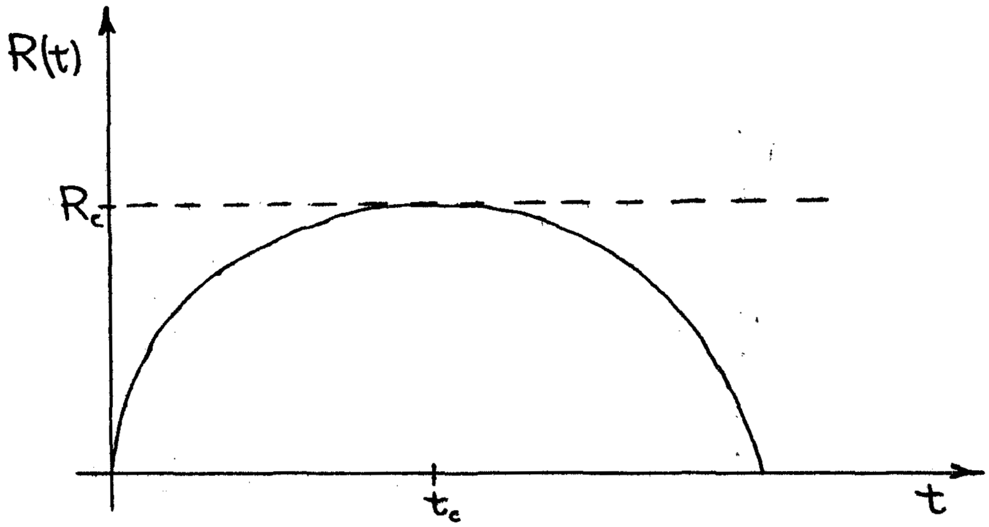


iii) $\Lambda < 0$

G(r) is a decreasing function of R, positive in $0 \leqslant R < R_c$ and negative for $R > R_c$, where R_c is the root of a cubic equation

$$R^3 - \frac{3k}{\Lambda} R + \frac{3C}{\Lambda} = 0$$

R_c is, of course, the real root. The expansion begins as before but slows down as $R \to R_c$ where $\frac{dR}{dt} = 0$. For $t > t_c$ the system contracts and runs through its previous phases until $R = 0$. Then the cycle begins again. This universe is an oscillating universe.



Case 2: $k = 0$

i) $\Lambda > 0$

G(R) is positive with a minimum at $R_m = \left(\frac{3C}{2\Lambda}\right)^{1/3}$ as in case 1 (i). This case is similar to that one except that we can obtain an exact solution.

$$\left(\frac{dR}{dt}\right)^2 = \frac{C}{R} + \frac{1}{3}\Lambda R^2$$

Since we expect this solution will give an infinite expansion choose the following type of solution

$$R^3 = A(\cosh Bt - 1)$$

$$3R^2 \frac{dR}{dt} = AB \sinh Bt$$

$$\left(\frac{dR}{dt}\right)^2 = \frac{(AB)^2 \sinh^2 Bt}{(3R^2)^2} = \frac{A^2B^2(\cosh^2 Bt - 1)}{9R^4} .$$

But

$$\cosh Bt = 1 + \frac{R^3}{A}$$

$$\left(\frac{d}{dt}\right)^2 = \frac{A^2B^2\left[\left(1 + \frac{R^3}{A}\right)^2 - 1\right]}{9R^4} = \frac{A^2B^2}{9R^4}\left(\frac{2R^3}{A} + \frac{R^6}{A^2}\right)$$

$$= \frac{2AB^2}{9R} + \frac{B^2}{9}R^2$$

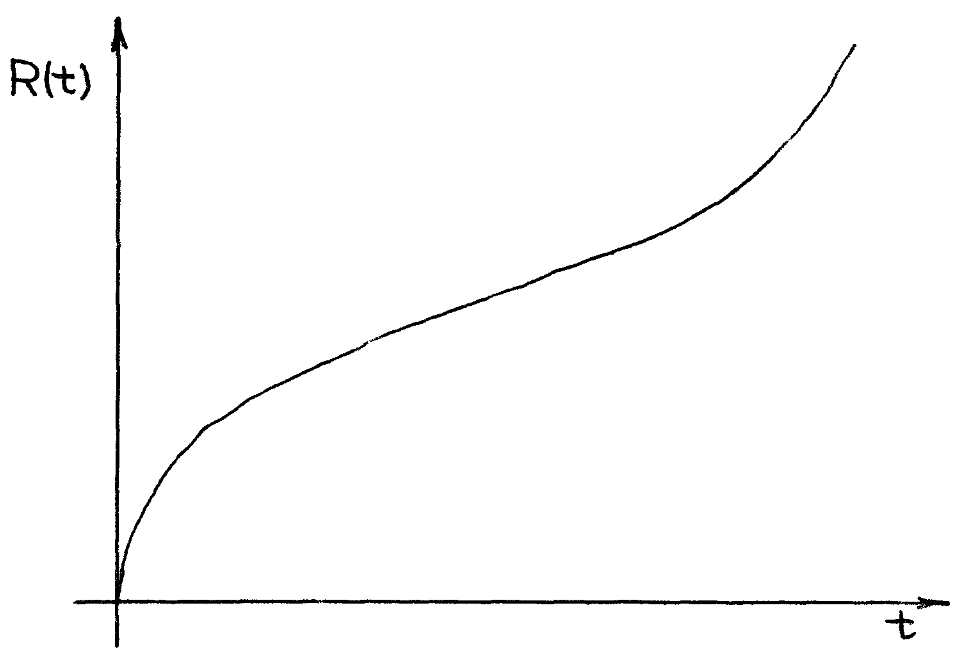
Hence

$$\frac{B^2}{9} = \frac{1}{3}\Lambda \qquad B = (3\Lambda)^{\frac{1}{2}}$$

$$\frac{2AB^2}{9} = C \qquad \frac{2A3\Lambda}{9} = C \Longrightarrow A = \frac{3C}{2\Lambda}$$

Finally

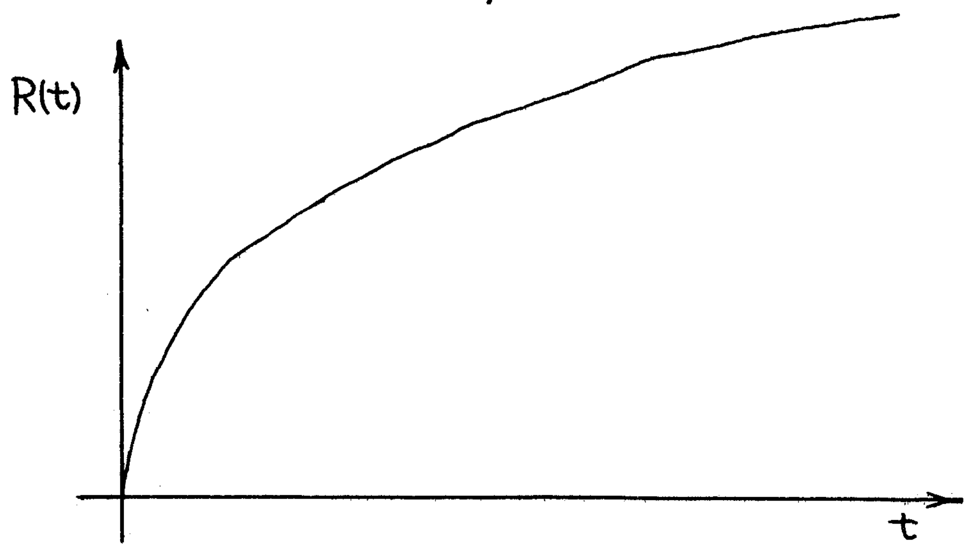
$$R^3 = \frac{3C}{2\Lambda}\left[\cosh(3\Lambda)^{\frac{1}{2}}t - 1\right]$$



ii) $\Lambda = 0$

This is a trivial example

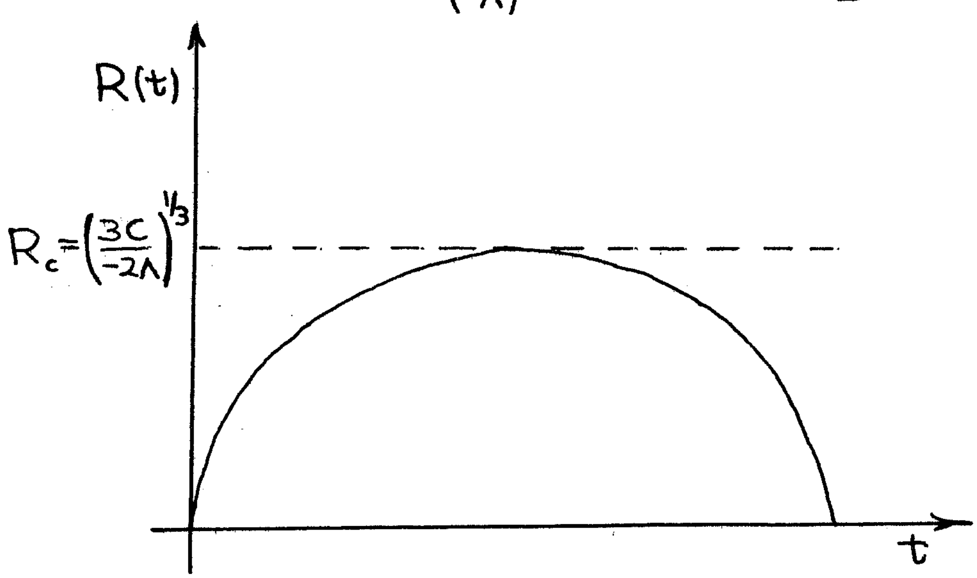
$$R = \left(\frac{9}{4}\, Ct^2\right)^{1/3}$$



iii) $\Lambda < 0$

This case is almost identical to case I(iii). Here the explicit solution is determined in the same manner as case II(i). One gets

$$R^3 = \frac{3C}{2(-\Lambda)}\left[1 - \cos\,(-3\Lambda)^{\frac{1}{2}}t\right]$$



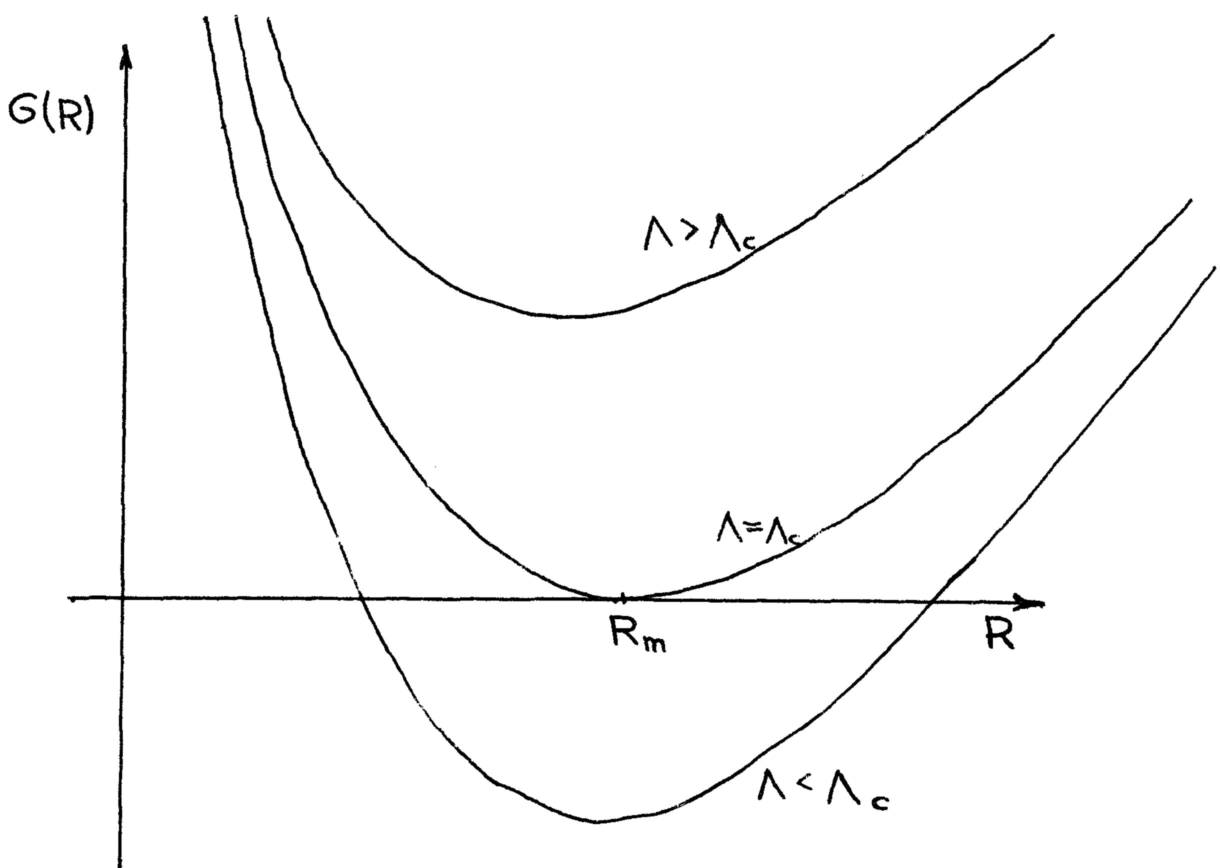
Case 3: $k>0$

This case exhibits much more variety than the previous cases. Since $k > 0$ $G(R)$ could be negative for appropriate values of k, Λ, and C. As in the previous cases $G(R)$ has a minimum as $R_m = \left(\frac{3C}{2\Lambda}\right)^{1/3}$. For what values of k, Λ, and C is the minumum of $G(R)$ equal to 0 ?

$$\frac{C}{R_m} - k + \frac{1}{3}\Lambda R_m^{\;2} = 0$$

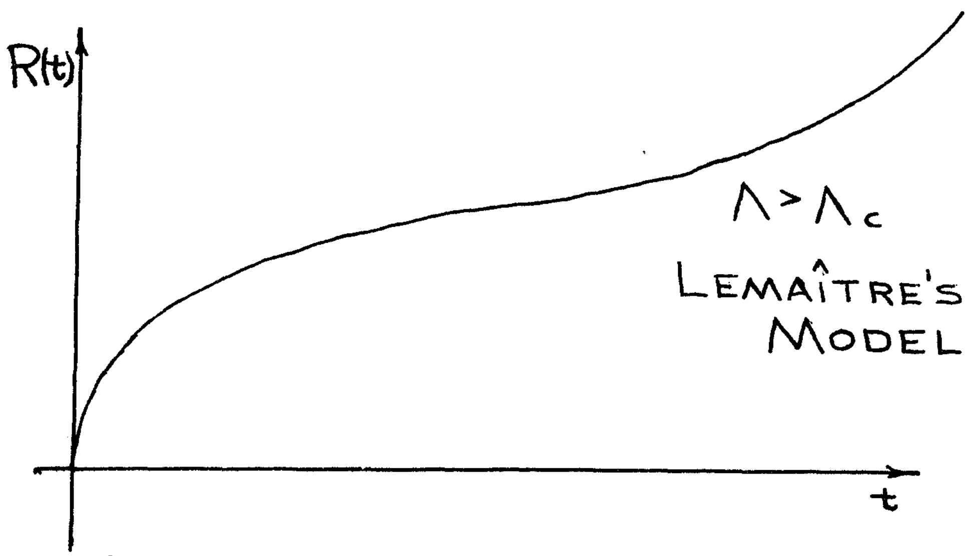
$$R_m k = C + \frac{\Lambda}{3}\,\frac{3C}{2\Lambda} \Rightarrow R_m k = \frac{3}{2}\,C$$

$$\frac{3C}{2\Lambda} = \frac{9.3}{4.2}\,C^3/k^3 \Rightarrow \Lambda_c = \frac{4k^3}{9C^2}$$



i) $\Lambda > \Lambda_c$

G(R) is always positive with a minimum at $R_m = \left(\frac{3C}{2\Lambda}\right)^{1/3}$. This case is similar to cases 1(i) and 2(i).



ii) $\Lambda = \Lambda_c$

G(R) is always positive except at the minimum point $R = \frac{3C}{2k} = R_c$. This gives rise to several possible solutions.

ii(a) There exists a static solution $R = R_c$.

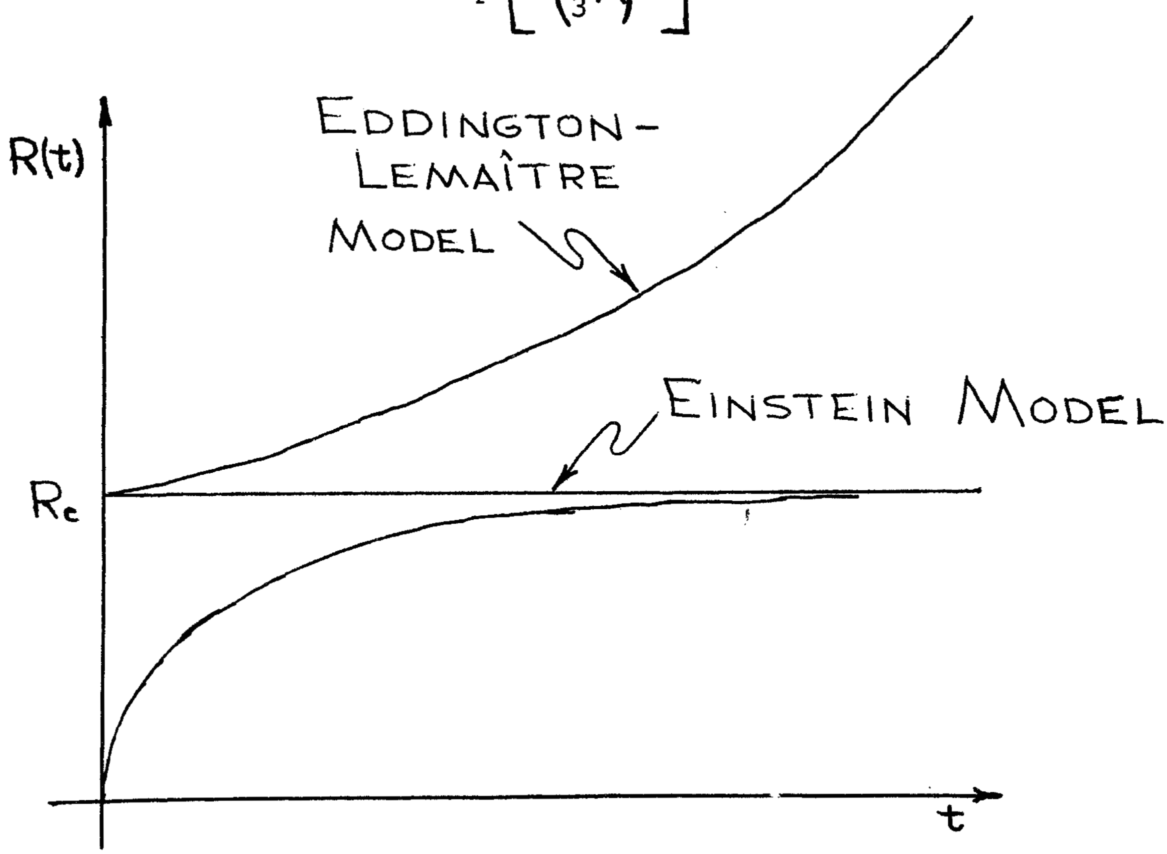
ii(b) For small times R behaves as case 1 and case 2.

$$R \sim \left(\frac{9Ct^2}{4}\right)^{1/3}$$

As usual the expansion will slow down as R increases. However in this case when $R = R_c$ the rate of expansion is 0 and the universe will become static.

ii(c) If $R > R_c$ originally the expansion will not be limited. As R gets large the only significant term will be $\frac{1}{3}\Lambda R^2$ so

$$R \sim \exp\left[t\left(\tfrac{1}{3}\Lambda\right)^{\frac{1}{2}}\right]$$



iii) $\Lambda_c > \Lambda > 0$

G(R) is positive for sufficiently large or small values of R. However there exists a range $R_1 < R < R_2$ where G(R) is negative. This gives rise to two solutions.

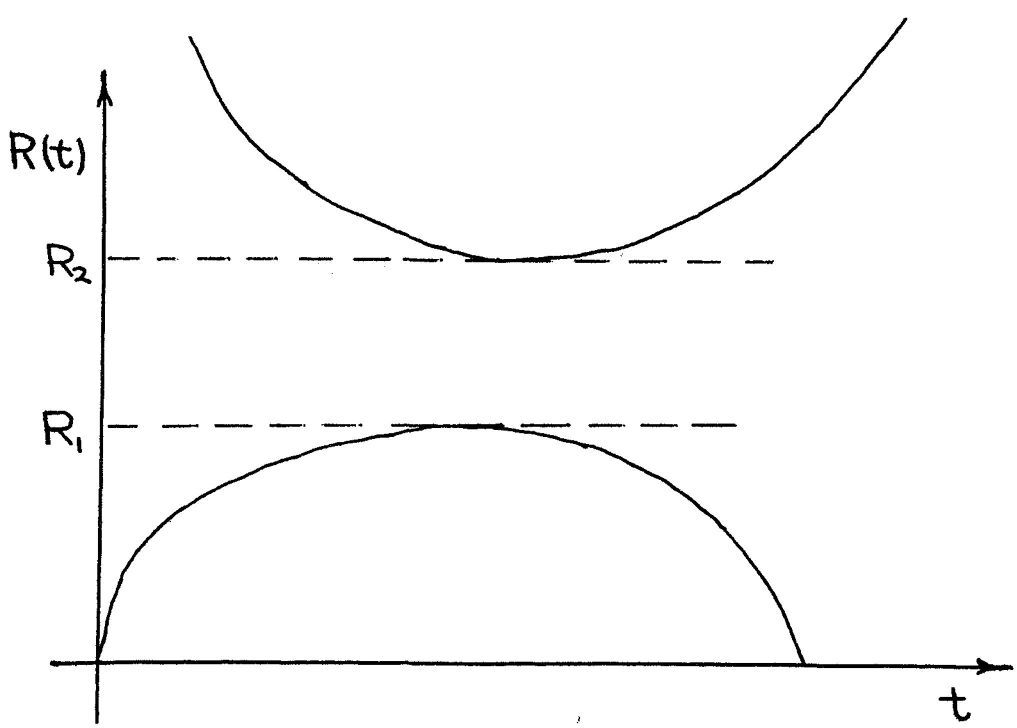
iii a) For this case $0 \leqslant R \leqslant R_1$, the solution is very similar to cases 1(iii) and 2(iii). The maximum value of R is R_1. As the universe approaches R_1 the expansion slows down, eventually reversing itself. This is an oscillating universe.

iii b) $R_2 \leqslant R$. Since R is never smaller than R_2 one need consider only the term in $\frac{1}{3}\Lambda R^2$. Then

$$R \sim \exp\left[t\left(\tfrac{1}{3}\Lambda\right)^{\frac{1}{2}}\right] \text{as} \quad t \to +\infty$$

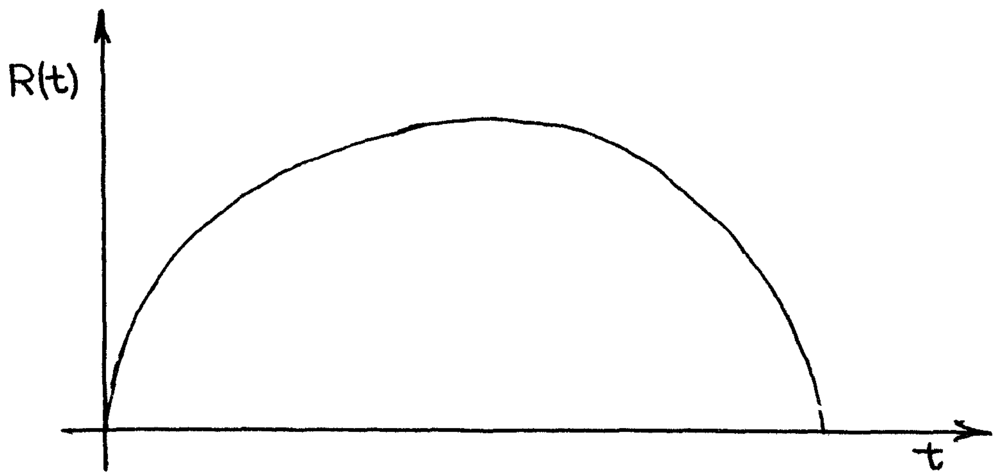
$$R \sim \exp\left[-t\left(\tfrac{1}{3}\Lambda\right)^{\frac{1}{2}}\right] \text{as} \quad t \to -\infty$$

Initially R decreases until the minimum radius R_2 is reached. After that the universe expands.



iv) $0 \geq \Lambda$

G(R) decreases monotonically. This case is equivalent to cases 1(iii) and 2(iii) and gives an oscillating universe.



These examples cover all possible cases. It is illuminating to classify them according to the type of universe they lead to.

Class I. The static, or Einstein universe: case 3(ii a).

Class II. Models which expand monotonically starting at a definite time from a point origin $R = 0$: cases 1(i), 1(ii), 2(i), 2(ii), 3(i).

Class III. The model which begins from a finite value of R at $t = -\infty$ with a gradually increasing expansion. This is the Eddington-Lemaître model: case 3(ii c).

Class IV. The model which starts at a finite time from a point origin $R = 0$ and expands more slowly as time increases. R tends to a finite limit as $t \to \infty$: case 3(ii b).

Class V. Models which oscillate between $R = 0$ and a finite value of R: cases 1(iii), 2(iii), 3(iii a), 3(iv).

Class VI. The model which contracts from infinite R at $t = -\infty$ to a finite minimum R and then expands to infinity: case 3 (iii b).

OBSERVABLE QUANTITIES - A practical view

The observational quantities that are available to distintuish between different cosmological models are:

1) apparent magnitude of galaxies
2) red shift
3) Angular diameters of galaxies and clusters of galaxies.
4) galaxy counts

Different cosmological models predict different relations for these quantities. In principle, therefore, observations should be able to determine the validity of these models. Unfortunately, however, the differences become significant only over distances of the order of the radius of curvature of the universe.

Also, the large variety of possible models makes any final decision impossible because enough observable parameters do not exist in the equations. Let us restrict our view then to those evolving models where the cosmological constant is zero. This reduces one parameter and makes a unique decision between the evolving models and the steady state model possible in principle.

Adopt the familiar assumptions of isotropy and homogeneity. The most general expression for a line element is given by

$$ds^2 = c^2\ dt^2 - R^2(t)\ du^2 = c^2\ dt^2 - R^2(t)\ \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2$$

du represents a three-space of constant Riemannian curvature which is independent of time. Two familiar examples are cartesian coordinates and spherical polar coordinates. The function R(t) is determined by introducing this line element into the Einstein field equations giving -

$$\frac{\dot{R}^2}{R^2} + \frac{2\ddot{R}}{R} + \frac{8\pi Gp}{c^2} = -\frac{k^2c^2}{R^2} + \Lambda c^2 \qquad (1)$$

$$\frac{\dot{R}^2}{R^2} - \frac{8\pi G\rho}{3} = -\frac{kc^2}{R^2} + \frac{\Lambda c^2}{3} \qquad (2)$$

P is the isotropic pressure of matter and radiation, ρ is the density of matter and energy, Λ is the cosmological constant and R^2 is the Riemann curvature. This curvature can be either greater than, less than, or equal to zero depending on the parameter k.

k =	+ 1	closed	Elliptic
=	- 1	open	Hyperbolic
=	0		Euclidean

Subtract Eq. (2) from Eq. (1), considering $\Lambda = 0$.

$$\frac{\ddot{R}}{R} + 4\pi G\left(\frac{\rho}{3} + \frac{P}{c^2}\right) = 0 \qquad (3)$$

Eq. (1) and Eq. (2) hold for all time. Denote the present values by a subscript 0 . The present value of the Hubble constant relates the velocity of recession with the distance.

$$H_o \equiv \frac{\dot{R}_o}{R_o}$$

Define a deceleration parameter q_o

$$q_o = -\frac{\ddot{R}_o}{R_o H_o^2}$$

Eq. (3) becomes:

$$\frac{\ddot{R}_o}{R_o} + 4\pi G\left(\frac{\rho_o}{3} + \frac{P_o}{c^2}\right) = 0 \qquad (4)$$

Substituting the deceleration parameter q_o

$$\rho_o + \frac{3P_o}{c^2} = \frac{3H_o^2 q_o}{4\pi G} \qquad (5)$$

Note that $q_o > 0$ for all physical systems. Assuming the present value of Hubble's constant $H_o = 75$ km/sec 10^6 pc we get

$$\rho_o + \frac{3P_o}{c^2} = 2.06 \times 10^{-29} q_o \text{ gm/cm}^3$$

If q_o can be determined from red shift measurements then $\rho_o + \frac{3P_o}{c^2}$ is known.

If the definition of Hubble's constant is substituted into Eq.(2) we get the following result for $\Lambda = 0$.

$$\frac{kc^2}{R_o^2} = \frac{8\pi G \rho_o}{3} - H_o^2$$

Combining this equation with Eq. (5)

$$\frac{kc^2}{R_o^2} = \frac{4\pi G}{3} 2\rho_o - \frac{4\pi G}{3q_o}\left(\rho_o + \frac{3P_o}{c^2}\right)$$

$$= \frac{4\pi G}{3q_o}\left(\rho_o(2q_o - 1) - \frac{3P_o}{c^2}\right) \tag{6}$$

This equation demonstrates a major tenet of general relativity - that the intrinsic geometry of space (k/R_o^2) is determined by the energy content of the universe as observed in the total density and pressure.

Let us estimate the values of density and pressure. The pressure comes from a radiation term $\frac{aT_o^4}{3}$ and a pressure due to the random galactic motion ($\rho_o V^2$) where V is the random radial velocity which is observed to be less than 300 km/sec. The density ρ_o comes from the matter density and the matter equivalent of the radiation density $\left(\frac{a T_o^4}{c^2}\right)$ where a is Stefan's constant.

Thus

$$\rho_{om} + \frac{2 a T_o^4}{c^2} + \frac{3 \rho_{om} V_o^2}{c^2} = \frac{3 H_o^2 q_o}{4 \pi G} \qquad (7)$$

At the present time the observed matter density is about 10^{-31} gm/cm^3. The radiation temperature of intergalactic space has recently been identified by several measurements as about 3° K. The radiation term $\frac{a T_o^4}{c^2}$ for 3° K background radiation is about 10^{-33} gm/cm^3. This is smaller than the observed matter density by two orders of magnitude. The random motion pressure term $\frac{3 \rho_{om} V^2}{c^2}$ is similarly negligible compared with ρ_{om}. It appears that at present P_o is approximately 0.

Then Eq. (5) and Eq. (6) become

$$\rho_o = \frac{3 H_o^2 q_o}{4 \pi G} = 2.06 \times 10^{-29} q_o \text{ gm/cm}^3$$

$$\frac{k c^2}{R_o^2} = H_o^2 (2 q_o - 1)$$

in a dust filled universe ($\rho \neq 0$, $P = 0$).

One observes that $q_0 \approx 1$. This predicts a density approximately 100 times the observed amount of matter. If the cosmological constant is zero, this may indicate the presence of large amounts of nonluminous matter in space.

The limits of q_0 for a dust filled universe are:

$q_0 > 1/2$	$k = +1$	closed, elliptical
$q_0 = 1/2$	$k = 0$	Euclidean
$0 \leq q_0 < 1/2$	$k = -1$	open, hyperbolic

the lower limit for exploding models is $q_0 = 0$ for $k = -1$. In this case $\rho_0 = 0$ and the universe is empty.

Look again at Eq. (7). As we have seen in the present epoch the universe has negligible pressure. However, if the temperature of the universe at some time in the past were significantly higher the pressure term would dominate. In an adiabatic expansion the radiation density decreases faster than the matter density ρ. General considerations indicate that T varies as $1/R$ while ρ varies as $1/R^3$. The radiation term varies as $1/R^4$ so for small R $\frac{aT^4}{c^2}$ is larger than ρ. Thus in the early stages of an exploding universe the radiation dominates over matter. At a time far enough in the past the entire density and pressure are effectively due to radiation alone.

$$\rho_o = \frac{U}{c^2} \qquad P = \frac{U}{3}$$

Then Eq. (7) becomes

$$2U_o = \frac{3H_o^2 q_o c^2}{4\pi G} \quad \text{where} \quad U = aT^4 \tag{8}$$

and Eq. (6) becomes

$$\frac{kc^2}{R_o^2} = \frac{8\pi G U_o}{3q_o c^2}(q_o - 1) \tag{9}$$

Substituting Eq. (8) into Eq. (9)

$$\frac{kc^2}{R_o^2} = H_o^2(q_o - 1)$$

The limits of q_0 for a radiation filled universe are:

$$q_0 > 1 \qquad k = +1$$
$$q_0 = 1 \qquad k = 0$$
$$0 < q_0 < 1 \qquad k = -1$$

A. MAGNITUDE - RED SHIFT RELATION

In 1928 Robertson and Hubble independently discovered a linear relation between apparent magnitude and red shift. Subsequent studies showed that the relation between log z and $m_{bol.}$ is linear for red shifts less than $z = 0.15$.

Theoretical calculations show that there should be a linear relation between the metric distance u and z for all models which obey the cosmological principle (isotropy, homogeneity). Unfortunately we are unable to directly observe metric distances. Only the apparent magnitude of stars can be directly measured with a telescope.

An additional complication arises which provides a possibility of distinguishing between cosmological models. The speed of light is finite so we observe different parts of the universe at different times. Thus if the expansion rate varies with time the relationship between $m_{bol.}$ and $\log z$ (red shift) should deviate from linearity. Of course no deviation will occur until the light travel time is great enough for a significant change in the expansion rate. Since the amount of deviation will depend on the different expansion rates this offers an obvious test of cosmological models.

The metric distance of a galaxy that emits photons at t_1 that are observed at t_o is

$$u = c\int_{t_1}^{t_0} \frac{dt}{R(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}} \qquad \text{where the observer is at } r = 0 \tag{10}$$

which follows from the general expression for a line element since for photons $ds = 0$.

Consider that the galaxy in question has total luminosity L. The apparent bolometric magnitude for the observer is

$$\ell = \frac{L}{4\pi R_o^2 \sigma^2 (1+z)^2} \tag{11}$$

$4\pi R_o^2 \sigma^2$ represents the area of the advancing radiation wave front. Because of the curvature of space this is not equal to $4\pi R^2 U^2$. Integrating the expression for the metric distance:

$$u = \begin{cases} \sin^{-1} r_1 & k = +1 \Rightarrow r_1 = \sin u \\ r_1 & k = 0 \Rightarrow r_1 = u \\ \sinh^{-1} r_1 & k = -1 \Rightarrow r_1 = \sinh u \end{cases}$$

The metric distance is $r_1 R_o$ or $R_o \sigma(u)$. Hence $\sigma(u) = r_1$.

The red shift can be expressed

$$1 + z = \frac{\nu_1 (\text{emitted})}{\nu_2 (\text{observed})} = \frac{R_o}{R_1} \tag{12}$$

These equations are now sufficient to obtain the $[m, z]$ relation.

In the steady state theory $k = 0$ so $\sigma(u) = u$. The expansion of the universe is required to be independent of time. Since $H_o = \frac{\dot{R}}{R}$ we have

$$R(t) = Be^{H_o t} \quad \text{where } B \text{ is constant} \tag{13}$$

Substitution of Eq. (13) into Eq. (10) gives

$$u = c\int_{t_1}^{t_o} \frac{dt}{R(t)} = \frac{c}{BH}\left(e^{-Ht_1} - e^{-Ht_o}\right) = \frac{c}{H}\left(\frac{1}{R_1} - \frac{1}{R_o}\right)$$

Substituting Eq. (12) into this equation we get:

$$u = \frac{c}{H_o}\left(\frac{1+z}{R_o} - \frac{1}{R_o}\right) = \frac{cz}{H_o R_o}$$

Eq. (11) now becomes

$$l_{bol.} = \frac{L H_o^2}{4\pi c^2 z^2 (1+z)^2}$$

which can be converted to apparent magnitude

$$m_{bol} = 5 \log z + 5 \log (1+z) + C$$

where C is a "constant" which depends on absolute luminosity.

Similar methods have been used to find $[m,z]$ relations for evolving models. It has been shown that for all models with $q_o > 0$

$$m_{bol} = 5 \log \frac{1}{q_o^2} \left[q_o z + (q_o - 1)\left(\sqrt{(1 + 2q_o z)} - 1\right) \right] + C \qquad (14)$$

For the case where $q_o = 0$

$$m_{bol} = 5 \log z \left(1 + \frac{1}{2} z\right) + C$$

The constant can be determined from observation. The part of the constant related to the red shift of distant galaxies can be split off as an additive factor k . The most practical wavelengths to observe are between 6200 Å and 7500 Å because for $z < \frac{1}{2}$ k_R is close to zero for the entire range of z . Then for the steady state model

$$m_R - k_R = 5 \log z (1+z) + 20.266 \qquad q_o = -1$$

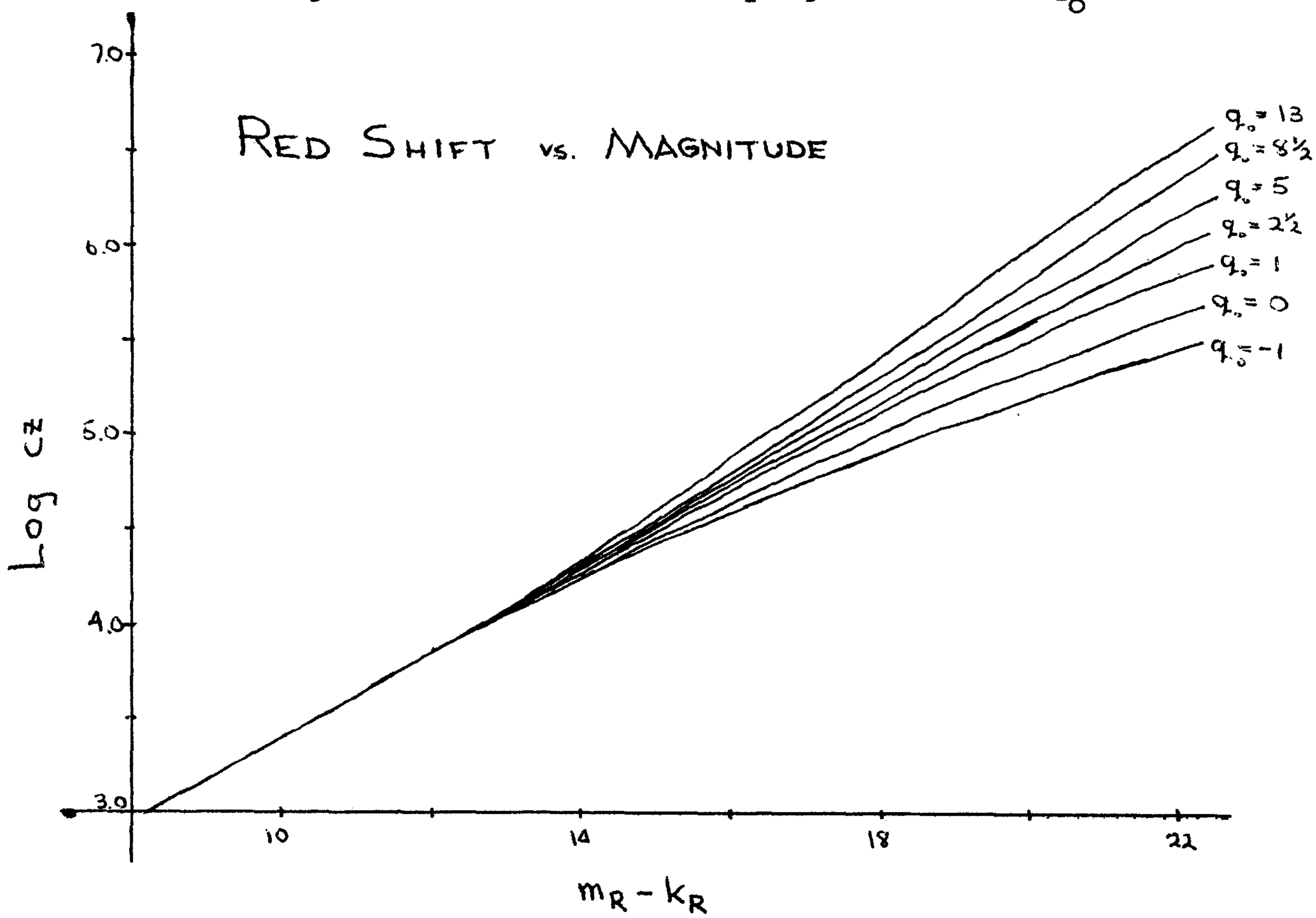
For an evolving model with $q_o > 0$ –

$$m_R - k_R = 5 \log \frac{1}{q_o^2}\left[q_o z + (q_o - 1)\left((1 + 2q_o z)^{1/2} - 1\right)\right] + 20.266$$

For the special case of $q_o = 0$

$$m_R - k_R = 5 \log z\,(1 + \frac{z}{2}) + 20.266$$

Plotting these results for varying values of q_o :



The 200 inch telescope can determine red shifts of $z = 0.5$ under perfect conditions. However, the accuracy of the magnitude determination is not sufficient to select any value for q_o. Photographic techniques must be improved and a more complete theory of galactic evolution must be developed before the observational data can be used to select the appropriate world model.

B. COUNT - MAGNITUDE RELATION

If galaxies are uniformly distributed in space counts to a distance u will be proportional to the volume enclosed within u. Volumes in Riemann space vary either faster or slower than U^3 according to whether $k = +1$ or -1 so a determination of the spatial curvature should be possible. In practice one counts galaxies to successive limits of magnitude where the relation between u and apparent luminosity is given by Eq. (11).

Let N(m) be the number of galaxies brighter than apparent magnitude m; n the number of galaxies per unit volume; Q the number of square degrees in the sky. Assume all galaxies have the same intrinsic luminosity. It has been shown (Mattig, AN., 284, 109) that

$$N(m) = \frac{2\pi n}{Q H_o^3} \begin{cases} (1-2q_o)^{-3/2}\left(P\sqrt{1+P^2} - \sinh^{-1} P\right) & \text{for } k=-1 \\ (2q_o-1)^{-3/2}\left(\sin^{-1} P - P\sqrt{1-P^2}\right) & \text{for } k=+1 \end{cases} \quad (15)$$

where

$$P = \frac{A\sqrt{k(2q_o-1)}}{q_o(1+A) - (q_o-1)(1+2A)^{1/2}}$$

and

$$A = 10^{0.2(m_R - k_R - c)} \tag{16}$$

For $k = 0$:

$$N(m) = \frac{4\pi n A^3}{3 Q H_0^3} \left[\frac{1}{2} \left(1 + A + \sqrt{1 + 2A} \right) \right]^{-3}$$

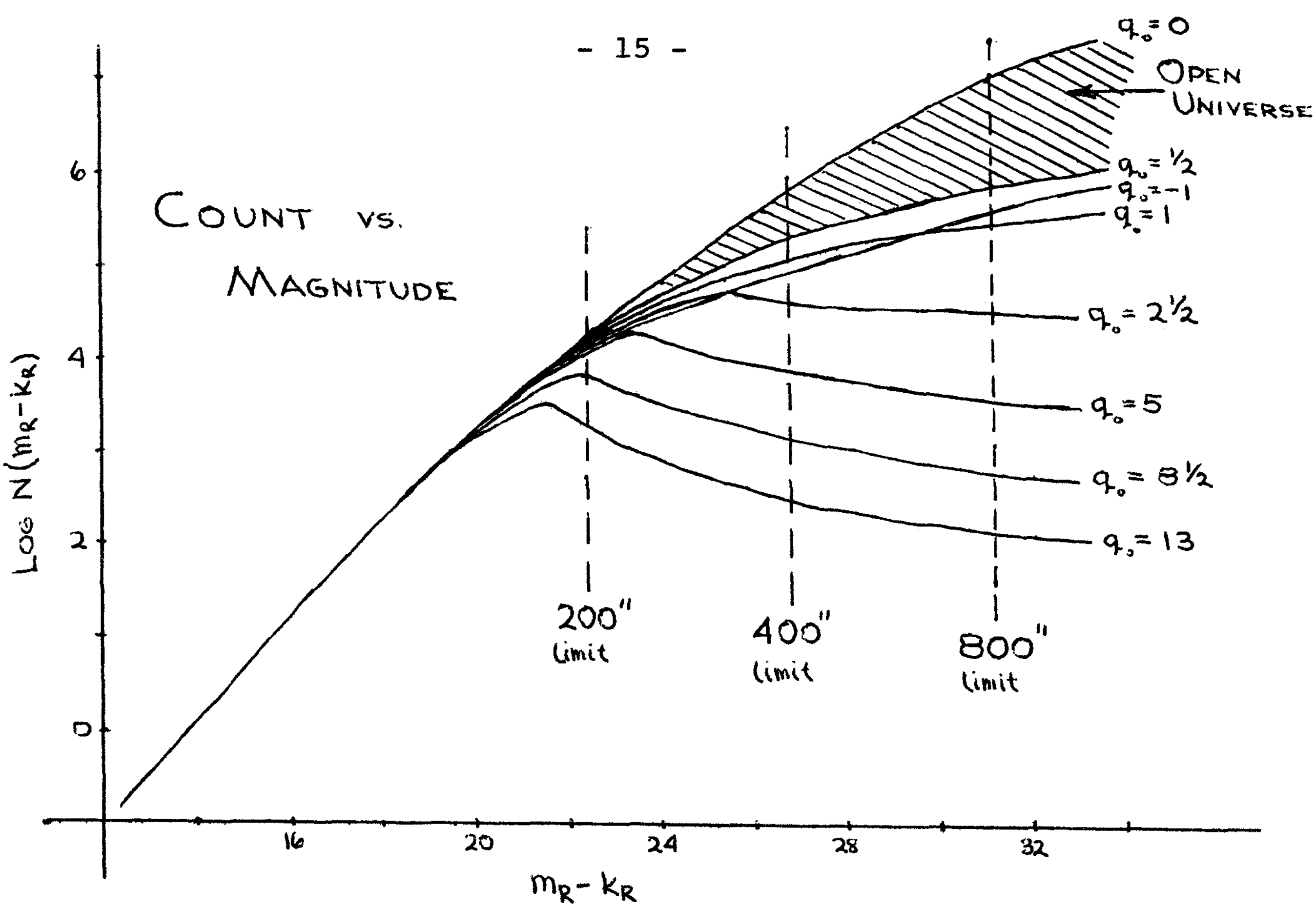
Bondi and Gold have given this in terms of the red shift

$$N(m) = \text{const.} \left[\ln (1 + z) - \frac{z(2 + 3z)}{2(1 + z)^2} \right]$$

For the observations we choose field galaxies rather than cluster galaxies so the constant in the $[m, z]$ relation will be different. One finds

$$m_R - k_R = 5 \log z + 22.516 \qquad z \ll 1$$

This system of equations can now be used to compute N(m) for various q_o values. The steady state model ($q_o = -1$) is compared with exploding models with $q_o = 0, \frac{1}{2}, 1, 2\frac{1}{2}, 5, 8\frac{1}{2}, 13$. These models represent a hyperbolic universe of zero density ($q_o = 0$), Euclidean space ($q_o = \frac{1}{2}$) and closed oscillating universes for $q_o > \frac{1}{2}$.



At first one is most amazed by this plot! How can the number of galaxies decrease as we look at fainter magnitudes? The behavior is explained by the periodicity of the equations. The opposite pole of the universe is reached when N(m) is a maximum and as u increases further one is really turned around and coming back. This situation is analogous to the surface of a sphere. When $r > \pi R$ the area begins to decrease from its maximum value of $4\pi R^2$. Hence the universe has already been counted by the time the maximum number of galaxies is reached.

Note that for models with $\frac{1}{2} < q_o < 1$ the antipole is not reached even though the universe is closed. Also the number of galaxies counted approaches an asymptotic limit for each model studied with $q_0 > 0$. This implies we have reached an observational horizon beyond which no information travels.

The observable differences at the limit of the 200" telescope are too small to provide any evidence in favor of a specific model.

C. ANGULAR DIAMETERS

Consider the standard line element:

$$ds^2 = c^2 dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

A source whose linear diameter is y at a metric distance $R_1 \sigma(u)$ will subtend an observed angle

$$\theta_0 = \frac{y}{R_1 \sigma(u)}$$

Using the definition of the red shift

$$R_1 = R_0/1+z$$

we get

$$\theta_0 = \frac{y(1+z)}{R_0\, \sigma(u)} \qquad (17)$$

From a previous analysis we know $\sigma(u) = r_1$. It has been shown by Mattig (A.N., 284, 109) that

$$r_1 = \sigma(u) = \frac{c}{R_o H_o q_o^2 (1+z)} \left[q_o z + (q_o - 1) \sqrt{(1 + 2q_o z)} \right]$$

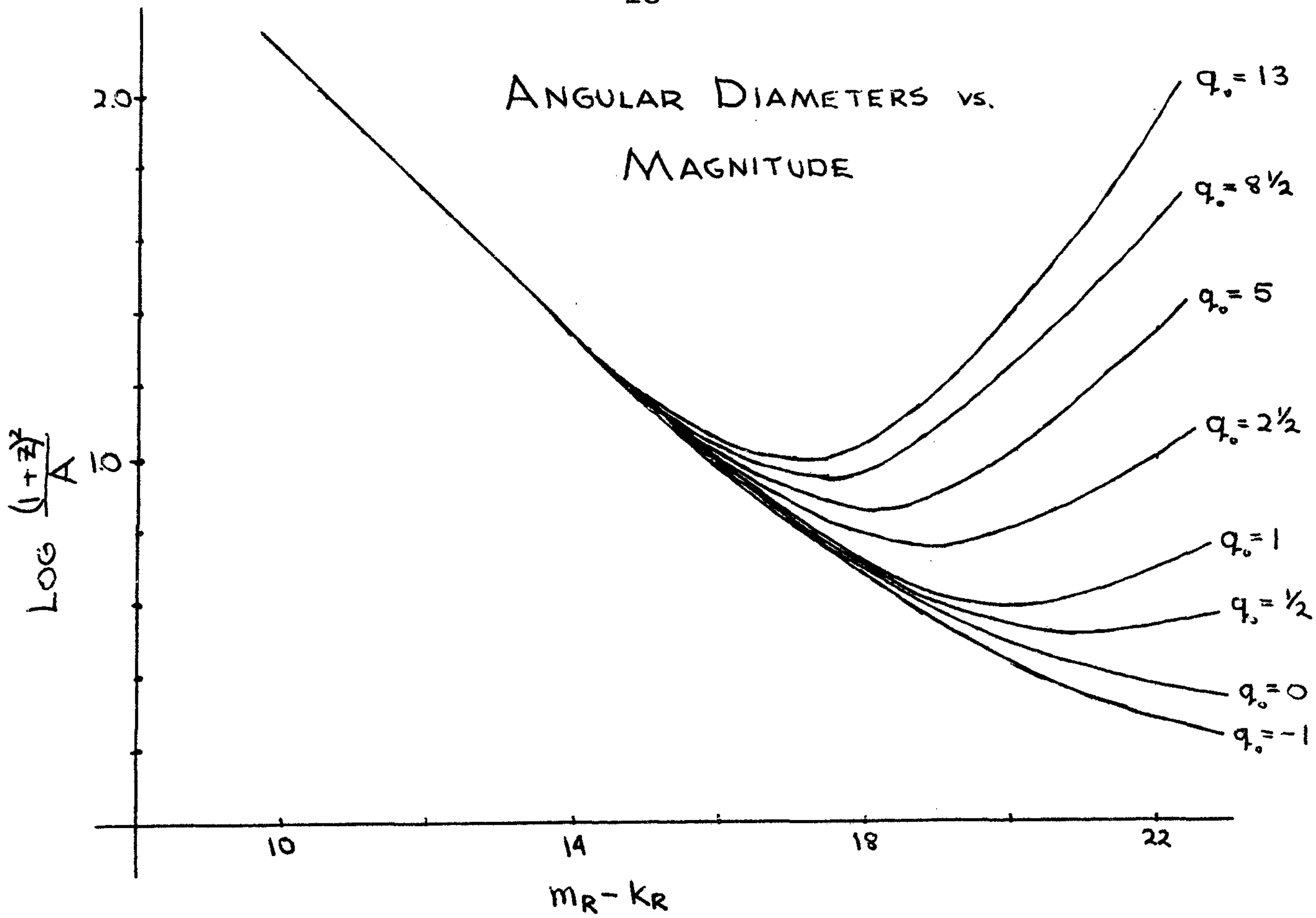
for all $q_o \geqslant 0$. Substituting this into Eq. (17) and using the definition of bolometric magnitude Eq. (14) and the definition of A Eq. (16) we get:

$$\Theta_o = \frac{\text{const.}\,(1+z)^2}{A} \qquad \text{for all } q_o \geqslant 0$$

For the steady state model $R_o\,\sigma(u) = R_o\,U = \frac{cz}{H_o}$ so

$$\Theta_o = \text{const.}\left(\frac{1+z}{z}\right)$$

Plot the apparent magnitude of galaxy clusters against their subtended angle.



Note that for all $q_o \geqslant 0$ the metric diameter decreases to a minimum at which point it begins to increase. This situation is analogous to the surface of a sphere. Standing at the "North Pole" one observes that a rod of constant length subtends a smaller angle as it is moved away from the pole. At the equator it substends the smallest possible angle and as it is moved toward the "South Pole" it begins to subtend greater and greater angles.

For the steady state model, however, Θ_o decreases asymptotically to the value of the const. as $z \rightarrow \infty$. This suggests that an experimental absence of a minimum would provide an acceptable test for the steady state theory. However, the minimum occurs beyond the range of the 200" telescope.